STOCHASTIC COMPARISON OF TESTS

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1. Introduction. It is shown in [1], in a special case, that the study (as random variables) of the levels attained when two alternative tests of the same hypothesis are applied to given data affords a method of comparing the performances of the tests in large samples. It is the object of the present paper to show that this method, which may be called stochastic comparison, is quite generally applicable. It is shown here, in particular, that in a given statistical context there is usually a wide class of tests such that, if test 1 and test 2 are in the class, the asymptotic efficiency of 1 relative to 2 is well defined and readily calculable. The argument is stated and discussed in general terms in Sections 2, 3 and 4, and illustrative examples are given in Section 5. The examples include comparison of the Wald-Wolfowitz test and the Smirnov test for two samples, and of the Kruskal-Wallis test and the F test for k samples.

2. Standard sequences. Consider an abstract sample space S of points s, and suppose that s is distributed in S according to some one of a given set $\{P_\theta\}$ of probability measures P_θ, where θ is an abstract parameter taking values in a set Ω. Let Ω_0 be a subset of Ω, and let H denote the hypothesis that $\theta \varepsilon \Omega_0$.

Let n be an index that takes the values $1, 2, 3, \cdots$. For each n, let T_n be a real valued statistic defined on S. We shall say that $\{T_n\}$ is a *standard sequence* (for testing H) if the following three conditions are satisfied.

I. There exists a continuous probability distribution function F such that, for each $\theta \varepsilon \Omega_0$,

$$\lim_{n\to\infty} P_\theta(T_n < x) = F(x) \quad \text{for every} \quad x. \tag{1}$$

II. There exists a constant a, $0 < a < \infty$, such that

$$\log\,[1 - F(x)] = -\frac{ax^2}{2}[1 + o(1)] \quad \text{as} \quad x \to \infty. \tag{2}$$

III. There exists a function b on $\Omega - \Omega_0$, with $0 < b < \infty$, such that, for each $\theta \varepsilon \Omega - \Omega_0$,

$$\lim_{n\to\infty} P_\theta\left(\left|\frac{T_n}{n^{\frac{1}{2}}} - b(\theta)\right| > x\right) = 0 \quad \text{for every} \quad x > 0. \tag{3}$$

The following is a typical example of a standard sequence. Let S be the set of all sequences $s = (x_1, x_2, \cdots \text{ad inf})$ with real coordinates x_n, let Ω be a set of distribution functions $\theta(x)$ on the real line such that $\mu(\theta) = \int_{-\infty}^{\infty} x\,d\theta \geqq 0$ and $\int_{-\infty}^{\infty} x^2\,d\theta < \infty$, and let P_θ denote the product measure $\theta \times \theta \times \cdots$ on S.

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