ON UNBIASED ESTIMATION[1]

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The theory of unbiased estimation has been mainly developed for quadratic loss-functions. The purpose of the present paper is to generalize this theory to convex loss-functions, and especially to loss-functions which are pth powers ($p \geqq 1$). The treatment of these cases needs in part quite different tools than in the quadratic case. Theorems of Stein and Bahadur are generalized. The contents of the paper have, however, some relations to results previously obtained by Barankin.

Let (R, S) be a measurable space and let $\mathfrak{P}$ be a nonempty class of probability measures P on S. Let g be any real valued function from $\mathfrak{P}$ into euclidean R_1. A real-valued measurable function on R for which $\int_R hdP$ exists for all $P \varepsilon \mathfrak{P}$ is called an unbiased estimator for g if

$$(1) \qquad E(h; P) = \int_R hdP = g(P),$$

for all $P \varepsilon \mathfrak{P}$.

The set of all h's which satisfy (1) will be designated by H_g. Let $\omega(z)$ be any nonnegative Borel-measurable function defined on $-\infty < z < \infty$. Denote by $H_g(\omega; P)$ the set of all $h \varepsilon H_g$ for which $E(\omega(h - g(P)); P)$ with $P \varepsilon \mathfrak{P}$ exists.

DEFINITION 1: $h_0 \varepsilon H_g(\omega; P_0)$ is called locally ω-minimal a $P_0 \varepsilon \mathfrak{P}$ if

$$E(\omega(h_0 - g(P_0));)P_0) \leqq E(\omega(h - g(P_0)); P_0)$$

for all $h \varepsilon H_g(\omega; P_0)$.

DEFINITION 2: $h_0 \varepsilon \bigcap_{P\varepsilon\mathfrak{P}} H_g(\omega; P)$ is called uniformly ω-minimal if

$$E(\omega(h_0 - g(P)); P) \leqq E(\omega(h - g(P)); P)$$

for all $h \varepsilon \bigcap_{P\varepsilon\mathfrak{P}} H_g(\omega; P)$ and every $P \varepsilon \mathfrak{P}$.

If $\omega(z)$ is of the form $|z|^p$, $p \geqq 1$, then we shall also use the phrase p-minimal instead of ω-minimal. The significance of $H_g(p; P)$ is obvious.

The case $\omega(z) = z^2$ is frequently treated in the literature. Only a few papers exist which are occupied with more general loss functions $\omega(z)$. I refer in this connection to investigations by Barankin [1].

We now give

DEFINITION 3. Let $V_p(p \geqq 1)$ be the class of all unbiased estimators v for $g \equiv 0$ such that $E(|v|^p; P)$ exists for all $P \varepsilon \mathfrak{P}$, and let $V_p^{P_0}$ be the class of all un-

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