ON THE INDEPENDENCE OF A SAMPLE CENTRAL MOMENT AND THE SAMPLE MEAN[1]

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1. Introduction. Let $X_1, X_2, \cdots, X_N$ be a random sample of size N (independently and identically distributed random variables) from a population with distribution function $F(x)$. It is known that the population can sometimes be characterized by the independence of a suitable statistic[2] $S = S(X_1, X_2, \cdots, X_N)$ and the sample mean $\bar{X} = \sum_{j=1}^N X_j/N$. If S is a polynomial statistic then the independence of S and $\bar{X}$ yields a differential equation for the characteristic function of $F(x)$. In order to determine $F(x)$ we must study this differential equation and find all its positive definite solutions. In the case of certain polynomial statistics, such as the k-statistics or quadratic polynomials, it is comparatively easy to obtain all positive definite solutions of this differential equation. In many cases however, this procedure is not feasible since it is often very difficult to decide whether a given function is positive definite. If we consider, for example, a normal population then any central sample moment $m_p = \sum_{j=1}^N (X_j - \bar{X})^p/N$ and the sample mean $\bar{X}$ are independent. But, when we investigate whether this property characterizes the normal population for $p > 3$, then it is practically impossible to determine all positive definite solutions of the corresponding differential equation.

In the present paper we prove the following theorem.

Theorem. *Let $X_1, X_2, \cdots, X_N$ be a sample of size N from a certain population. Let p be a positive integer such that $(p - 1)!$ is not divisible by $N - 1$. The population is normal if and only if the sample central moment m_p of order p is distributed independently of the sample mean $\bar{X}$.*

Remark. The condition that $(p - 1)!$ is not divisible by $N - 1$ is satisfied if $N > (p - 1)! + 1$.

For the proof of this statement we use a theorem which was recently derived by Linnik [1] and Zinger [2].

In Section 2 we derive two combinatorial lemmas which are essential for the proof of the theorem. In Section 3 we give some analytical results and deduce finally the theorem in Section 4.

2. Combinatorial lemmas. Let $x_0, x_1, \cdots, x_n$ be $n + 1$ real variables. Suppose that

$$P = P(x_0, x_1, \cdots, x_n) = \sum{}^{*} A_{j_0 j_1 \cdots j_n} x_0^{j_0} x_1^{j_1} \cdots x_n^{j_n} \tag{2.1}$$

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[2] A statistic is a real, single valued and measurable function of the observations $X_1, X_2, \cdots, X_N$.