EQUALITIES FOR STATIONARY PROCESSES SIMILAR TO AN EQUALITY OF WALD

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I. Introduction. Let Ω be a non-empty set with elements ω, $\mathfrak{F}$ be a σ-algebra of subsets of Ω and P be a probability measure on $\mathfrak{F}$. Let T be a one to one map of Ω onto Ω which, together with its inverse T^{-1} are $\mathfrak{F}$-measurable and P measure preserving. For any random variable (real $\mathfrak{F}$-measurable function) X on Ω, let TX be the function on Ω defined by $TX(\omega) = X(T\omega)$ so that $[TX \varepsilon B] = T^{-1}[X \varepsilon B]$ for any Borel set B. Consider an $\mathfrak{F}$-measurable set E with $P(E) > 0$. For any $\omega \varepsilon E$ consider the images of ω under iterates of T: $T\omega$, $T^2\omega$, $\cdots$, $T^n\omega$, $\cdots$. If n_1 is the smallest positive integer for which $T^{n_1}\omega \varepsilon E$ we say that the first recurrence time ν_1 of E is equal to n_1. The Poincaré recurrence theorem ([2], p. 10) asserts that ν_1 is well defined and finite almost everywhere on E. In fact the stronger version of the Poincaré recurrence theorem asserts that, for almost all $\omega \varepsilon E$, there are infinitely many positive integers n such that $T^n\omega \varepsilon E$. Let us write down these integers according to their natural order, n_1, $n_1 + n_2$, $n_1 + n_2 + n_3$, $\cdots$. Then n_k is defined to be the value of the kth recurrence time ν_k of ω. Thus the successive recurrence times of E: ν_1, ν_2, $\cdots$ are well defined almost everywhere on E. If we introduce the conditional probability measure given E, P_E, on $\mathfrak{F}$ by

$$P_E(A) = P(E \cap A)/P(E), \tag{1}$$

then ν_1, ν_2, $\cdots$ are well defined and finite valued with P_E probability one on the whole space Ω. In [3] it was proved that $\{\nu_k\}$ is a stationary sequence under P_E measure. In this paper we shall introduce a P_E measure preserving transformation S which associates with $\{\nu_k\}$ in a very natural way. It is shown that $S^{k-1}\nu_1 = \nu_k$, $k = 1, 2, \cdots$, so that the stationarity of $\{\nu_k\}$ is actually due to the P_E measure preserving property of S. Let $X_n = T^nX$. It is then shown that sequences $\{X_{\nu_k}\}$ and $\{X_{\nu_1+\cdots+\nu_{k-1}+1} + \cdots + X_{\nu_1+\cdots+\nu_k}\}$ are stationary under P_E measure. This leads to equalities (13) and (15), which resemble an equality of Wald for an independent sequence of random variables [1]. In fact, the proofs of (13) and (15) are also rather similar to the proof given in [1].

II. The Transformation S. Let $\bar{E}$, $\underline{E}$ be subsets of Ω defined by

$$\bar{E} = E \cap \left(\bigcup_{n=1}^{\infty} T^{-n}E\right), \tag{2}$$

$$\underline{E} = E \cap \left(\bigcup_{n=1}^{\infty} T^{n}E\right). \tag{3}$$

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