ON THE GENERAL TIME DEPENDENT QUEUE WITH A SINGLE SERVER

By J. Keilson and A. Kooharian

Sylvania Applied Research Laboratory

1. Introduction. In a previous paper [1] we discussed a multi-dimensional phase space model of queuing processes. The approach was developed in detail there for a time dependent queue subject to Poisson arrival and general service time distributions. The present paper extends this approach to the study of a time dependent queue with a single server subject to general arrival and general service time distributions. For this case we are able to carry out the analysis in detail in terms of the state densities introduced in [1]. The problem leads to simultaneous Wiener-Hopf equations with an analytic side condition. We resolve the problem by establishing its equivalence to a Hilbert problem (in the sense of [2]), for which we can give an explicit solution.

The analysis of the time dependent problem is valid whether the system tends to an equilibrium state or not. Thus we are able to derive expressions for the system regeneration time and server occupation time distributions which are valid for unstable as well as stable queues (Section 7).

A brief outline of the paper follows. In Sections 2 and 3 we describe the appropriate phase space for the system and develop the corresponding differential equations, boundary conditions and initial conditions for the general time dependent problem. The solution of this problem is based on the analysis of an associated "first passage" problem which we formulate in Section 4. In Section 5 we take advantage of the essential Wiener-Hopf character of the problem in order to reduce it to an integral equation. In Sections 6 and 7 we show how to formulate an equivalent homogeneous Hilbert problem for which we give an explicit solution. Next in Section 8 we generalize the first passage problem for arbitrary initial conditions thereby obtaining an associated inhomogeneous Hilbert problem whose solution is shown to be intimately related to that of the preceding homogeneous problem. Finally in Section 9 we show how the results from the first passage problems can be used to obtain the complete solution of our original general queuing problem of Section 3. In the concluding Section 10, we point out some connections between this work and that of Lindley [3] and Pollaczek [4].

A few remarks are in order here concerning the nature of the arguments we present in deriving the basic equations describing our process in Sections 2, 3, and 4. The basic quantities we work with are the *densities* of the time dependent probability distributions over the state space. It is not *a priori* evident that these densities exist, let alone possess the requisite smoothness properties for the derivations in Sections 3 and 4. Our justification for the arguments in these

Received July 19, 1960; revised October 31, 1961.