MERGING OF OPINIONS WITH INCREASING INFORMATION[1]

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1. History. One of us [1] has shown that if Zn, $n = 1, 2, \cdots$ is a stochastic process with D states, $0, 1, \cdots, D - 1$ such that $X = \sum_{n=1}^{\infty} Z_n/D^n$ has an absolutely continuous distribution with respect to Lebesgue measure, then the conditional distribution of $R_k = \sum_{n=1}^{\infty} Z_{k+n}/D^n$ given $Z_1, \cdots, Z_k$ converges with probability one as $k \to \infty$ to the uniform distribution on the unit interval, in the sense that for each λ, $0 < \lambda \leqq 1$, $P(R_k < \lambda \mid Z_1, \cdots, Z_k) \to \lambda$ with probability 1 as $k \to \infty$. It follows that the unconditional distribution of R_k converges to the uniform distribution as $k \to \infty$. If $\{Z_n\}$ is stationary, the distribution of R_k is independent of k, and hence uniform, a result obtained earlier by Harris [3]. Earlier work relevant to convergence of opinion can be found in [4, Chap. 3, Sect. 6].

Here we generalize these results and also show that the conditional distribution of R_k given $Z_1, \cdots, Z_k$ converges in a much stronger sense. All probabilities in this paper are countably additive.

2. Statement of the theorem. Let $\mathcal{B}_i$ be a σ-field of subsets of a set X_i, $i = 1, 2, \cdots$; and let $(X, \mathcal{B}) = (X_1 \times X_2 \times \cdots, \mathcal{B}_1 \times \mathcal{B}_2 \times \cdots)$. Suppose $(X, \mathcal{B}, P)$ is a probability space and let P_n be the marginal distribution of $(X_1 \times \cdots \times X_n, \mathcal{B}_1 \times \cdots \times \mathcal{B}_n)$; that is, $P_n(A) = P(A \times X_{n+1} \times \cdots)$ for all $A \in \mathcal{B}_1 \times \cdots \times \mathcal{B}_n$. The probability P is *predictive* if for every $n \geqq 1$, there exists a *conditional distribution* P^n for the *future* $X_{n+1} \times \cdots$ *given the past* $X_1, \cdots, X_n$; that is, if there exists a function $P^n(x_1, \cdots, x_n)(C)$ where $(x_1, \cdots, x_n)$ ranges over $X_1 \times \cdots \times X_n$ and C ranges over $\mathcal{B}_{n+1} \times \cdots$ with the usual three properties: $P^n(x_1, \cdots, x_n)(C)$ is $\mathcal{B}_1 \times \cdots \times \mathcal{B}_n$-measurable for fixed C; a probability distribution on $(X_{n+1} \times \cdots; \mathcal{B}_{n+1} \times \cdots)$ for fixed $(x_1, \cdots, x_n)$; and for bounded $\mathcal{B}$-measurable ϕ

$$\int \phi \, dP = \int [(\phi(x_1, \cdots, x_n, x_{n+1}, \cdots) \, dP^n (x_{n+1}, \cdots \mid x_1, \cdots, x_n)] \cdot dP_n (x_1, \cdots, x_n) \tag{1}$$

holds.

The assumption that P is predictive is mild and applies to all natural probabilities known to us. It is easy to verify that any probability which is absolutely continuous with respect to a predictive probability is also predictive.

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