of using simply $\phi(x)$, $x$ a single observation from the uniform distribution, one should use,

$$\{\phi(x) + \phi(1 - x) + \phi(y)\}/3$$

where

$$\begin{aligned} y &= x & 0 \leqq x \leqq \tfrac{1}{2} \\ &= \tfrac{3}{2} - x & \tfrac{1}{2} < x \leqq 1 \end{aligned}$$

for example. The reduced variance property of this estimate is a result of the above theorem. $G$ consists of the identity transformation, the transformation $x \to 1 - x$, and the transformation $x \to y$. Each of these transformations then has weight $\frac{1}{3}$.

## REFERENCES

[1] HODGES, J. L., JR. and LEHMANN, E. L. (1950). Some problems in minimax point estimation. *Ann. Math. Statist.* **21** 182–197.

[2] HAMMERSLEY, J. M. and MORTON, K. W. (1956). A new Monte Carlo technique, antithetic variates. *Proc. Cambridge Philos. Soc.* **52** 449–475.

# ON STOCHASTIC APPROXIMATIONS

BY SHMUEL FRIEDMAN

*The Hebrew University, Jerusalem*

**0. Summary.** The procedure of stochastic approximations suggested by Robbins-Monro [1], for reaching a zero point $x_0$ of a regression function, was shown by Dvoretzky [4], to be a convergent w. p. 1. and in mean square under certain conditions. In this paper we deal with two problems of modifying the process to acquire convergence under weaker conditions.

**1. Introduction.** Let $H(y/x)$ be a family of distribution functions, which correspond to the parameter $x$.

Let us write: $m(x) = \int y\, dH(y/x)$; $\sigma^2(x) = \int (y - m(x))^2\, dH(y/x)$.

Let $\{a_n\}$ be a sequence of positive members, such that, $\sum a_n = \infty$, $\sum a_n^2 < \infty$.

Let $x_1$ be an arbitrary number. The Robbins-Monro process is defined recursively for all $n$ by $x_{n+1} = x_n - a_n y_n$, where $y_n$ is a chance variable with distribution function $H(y/x_n)$. The conditions for its convergence were shown to be:

$$|m(x)| \leqq L\,|x| + K. \tag{1}$$

$$\sigma^2(x) \leqq \sigma^2 < \infty. \tag{2}$$

$$\text{If} \quad x < x_0, \quad \text{then} \quad m(x) < 0, \quad \text{while if} \quad x > x_0, \quad \text{then} \quad m(x) > 0. \tag{3}$$

Received August 28, 1961; revised January 1, 1962.

Institute of Mathematical Statistics is collaborating with JSTOR to digitize, preserve, and extend access to *The Annals of Mathematical Statistics.* www.jstor.org