OPTIMUM ESTIMATORS OF THE PARAMETERS OF NEGATIVE EXPONENTIAL DISTRIBUTIONS FROM ONE OR TWO ORDER STATISTICS

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1. Introduction and summary. Let

$$f_1(x) = \sigma^{-1} \exp(-x/\sigma), \text{ if } x \geqq 0; 0, \text{ otherwise};$$

$$f_2(x) = \sigma^{-1} \exp[-(x - \alpha)/\sigma], \text{ if } x \geqq \alpha; 0, \text{ otherwise}.$$

Let x_k denote the kth order statistic of a random sample of size n. Harter [1] discusses the following three problems designated here as P_1, P_2, and P_3:

P_1 : Best unbiased estimator of the form $c_k x_k$ for σ of $f_1(x)$;

P_2 : Best unbiased estimator of the form $c_l x_l + c_m x_m$ for σ of $f_1(x)$;

P_3 : Best unbiased estimators of the form $c_l x_l + c_m x_m$ for σ, α, and the mean, μ, of $f_2(x)$. For P_3 he shows that the optimum l is equal to 1 and that the same m is optimum for all three parameters. In each problem, after setting up the equation for the relative efficiency of a linear combination of one or two order statistics, he remarks that he is not aware of any analytical method for determining the best combination, and hence finds them by exhaustive numerical computations for n up to 100. In this paper an analytical method for his problems will be presented. For P_1 and P_3 the correct optimum values of k and m are readily determined *for all n*. These will be given in Sections 2 and 3. The equations for P_2, however, are quite difficult to solve. The analytical formulation of P_2 and an approximate solution, arrived at by trial and error, will be presented in Section 4.

The method is based on the Euler-Maclaurin formula

$$\sum_{r=0}^{k-1} f(r) = \int_0^k f(x)\,dx - \frac{1}{2}[f(k) - f(0)] + \left(\frac{1}{12}\right)[f^{(1)}(k) - f^{(1)}(0)] - \left(\frac{1}{720}\right)[f^{(3)}(k) - f^{(3)}(0)] + \cdots. \tag{1.1}$$

For a discussion of the remainder after a finite number of terms on the right we refer to [2]. It is sufficient to note here that this is an asymptotic expansion and the most accurate result is obtained by taking the sum to one-half of the smallest term.

2. Estimating σ of $f_1(x)$ from one x_k. $c_k x_k$ is an unbiased estimator of σ, where [1]

$$c_k = 1/\sum_1^k a_i, \qquad a_i = 1/(n - i + 1), \tag{2.1}$$

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