PROBABILITY INTEGRALS OF MULTIVARIATE NORMAL AND MULTIVARIATE t[1]

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1. Introduction. The evaluation of multivariate normal probability integrals is of special importance to the statistician dealing with multivariate problems. The joint distribution of several dependent continuous variates is often assumed to be a multivariate normal distribution, and the multivariate normal distribution also provides an approximation to the multinomial distribution for a large sample size. Applications of the multivariate normal distribution are numerous throughout statistical literature. An extensive list of applications of the bivariate normal probability distribution is given by D. B. Owen [8].[3] The present paper gives a survey of the work on multivariate probability integral and related functions starting with the bivariate case and includes the author's recent work on the probability integrals of the multivariate normal and a multivariate analogue of Student's t. Two new tables of the probability integrals of the equicorrelated multivariate normal are given at the end of the paper (Tables I and II).

2. Bivariate normal integral. W. F. Sheppard was perhaps the first statistician who concerned himself with the evaluation of the bivariate probability integral. In his paper [11] published in 1900, Sheppard obtained the exact probability that both the standardized correlated normal variates are positive and also discussed the calculation of certain V functions for evaluating the probability integral in general. In 1901, Pearson [31] published a method for evaluating the integral as a power series in ρ involving tetrachoric functions. The function computed by Pearson and his associates is:

$$\frac{d}{N} = \int_h^\infty \int_k^\infty f(x, y; \rho)\, dx\, dy \tag{1}$$

where

$$f(x, y; \rho) = [1/2\pi(1 - \rho^2)^{\frac{1}{2}}] \exp [-\tfrac{1}{2}\{x^2 - 2\rho xy + y^2\}/(1 - \rho^2)].$$

Pearson obtained the following tetrachoric series expansion:

$$d/N = \sum_{j=0}^{\infty} \rho^j \tau_j(h) \tau_j(k) \tag{2}$$

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[3] The numbers in brackets [ ] refer to the separate bibliography on the multivariate normal integrals and related topics which follow this paper.