LIMITING DISTRIBUTION OF THE MAXIMUM OF A DIFFUSION PROCESS[1]

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1. Introduction. Let $X(t)$, $t \geqq 0$ be a strong homogeneous Markov process on the interval of real numbers (r_1, r_2), $-\infty \leqq r_1 < r_2 \leqq \infty$, with continuous sample functions. For $t > 0$, let $Z(t)$ be the maximum value attained by $X(s)$ on the interval $[0, t]$: $Z(t) = \max\{X(s); 0 \leqq s \leqq t\}$. In this paper we shall investigate the limiting distribution of $Z(t)$ as $t \to \infty$ for several general types of Markov processes.

First we consider a process having a finite expected first passage time between every pair of points in (r_1, r_2). For this process it is known that a stationary distribution exists [14]; many limit theorems which are valid for sequences of independent random variables also hold for this process [17]. We use the well known renewal principle to show that the asymptotic behavior of $Z(t)$ is similar to that of the maximum in a sequence of independent, identically distributed random variables. Our results are applied to a process whose transition probability function satisfies the classical backward diffusion equation. An analytic method of getting the limiting distribution of $Z(t)$ from asymptotic estimates of the solution to the Fokker-Planck equation has been given by Newell [15]; his results are very close to special cases of our Theorem 5.1.

For certain processes we cannot find the limiting distribution of $Z(t)$ but can assert some form of asymptotic stability such as

$$\lim_{t\to\infty} Z(t)/c(t) = 1$$

in probability for some real function $c(t) \to \infty$. We use the theory developed by Gnedenko [10] and Geffroy [9] for the stability of the maximum in sequences of independent random variables. Similar results have been obtained for stationary normal processes by Cramér [4] and the writer [2].

The above theory is in the spirit of the extension of classical "extreme value" methods [11] to dependent random variables [3], [18]. In the last part of this study, we consider an entirely different type of process, for which "extreme value" methods do not work. We unveil an analogy between the distribution of $Z(t)$ and a distribution arising in renewal theory [8].

In some of the proofs of our results, we shall employ certain fundamental relations for recurrent diffusion processes due to Maruyama and Tanaka [14].

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