AN APPROXIMATION TO THE DISTRIBUTION OF Q (A VARIATE RELATED TO THE NON-CENTRAL t)[1]

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1. Introduction. If W is a random normal variate with mean θ and variance 1 and Z^2 is independently distributed as chi-squared with n degrees of freedom, then the random variable Q, with non-centrality θ and n degrees of freedom, is defined by

$$Q = W/(W^2 + Z^2)^{\frac{1}{2}}, \qquad \text{(all square roots positive)}. \tag{1}$$

Properties of Q, including the probability density function and the moments, have been studied elsewhere by Hogben et al. (1964).

It is apparent that $n^{\frac{1}{2}}Q/(1 - Q^2)^{\frac{1}{2}}$ is distributed as non-central t with non-centrality θ and n degrees of freedom. For some discussion of the non-central t variate see for example Fisher (1931), Johnson and Welch (1940) and Resnikoff and Lieberman (1957). The probability integral of Q is of course implicitly defined by that of the non-central t, but the tables of the non-central t of Resnikoff and Liberman provide no information for small values of θ. The direct definition of the distribution function of Q in terms of the density is not conveniently manageable except with extensive tabulation.

Miss van Eeden (1961) has compared six types of approximation for the non-central t-distribution by tabling exact and approximate percentage points for several probability levels, degrees of freedom and non-centrality values. Because percentage points are used, further numerical study would be necessary to compare her results with the results in Table 1. None of the approximations considered by Miss van Eeden has the correct limiting distribution when $\theta = 0$.

The purpose here is to propose as an approximation to the distribution of Q that of a linearly transformed beta variate and to present some results on the adequacy of the approximation. The approximation, which is especially good for small θ, implicitly and easily yields approximate values for the non-central t. The present interest in the random variate Q arose from its relevance in another study, Hogben et al. (1962).

2. An approximation. Let X be distributed as beta (see for example Mood (1950)) with parameters a and b. Then, the suggested approximation to the

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