

ON ADDING INDEPENDENT STOCHASTIC PROCESSES

BY TOM S. PITCHER

Lincoln Laboratory,¹ Massachusetts Institute of Technology

Let x be a stochastic process on an interval T , P_x be the probability measure it induces on the space Ω of sample functions on T , and M be the set of functions f in Ω such that P_{x+f} is absolutely continuous with respect to P_x (written $P_{x+f} < P_x$). The functions in M are "at least as smooth as" the sample functions of x in the sense that any smoothness property possessed by almost all of the sample functions must also be possessed by each function in M . Similarly, if y is a process independent of x such that $P_{x+y} < P_x$, then almost all the sample functions of y must possess any smoothness property possessed by almost all the sample functions of x . This might lead one to conjecture that $P_{x+y} < P_x$ would imply $P_y(M) = 1$ but this fails to hold even in the Gaussian case [1]. What is true in the Gaussian case is that $P_y(M) = 1$ if and only if the measure Q associated with the vector process $(x + y, y)$ is absolutely continuous with respect to the measure P associated with the process (x, y) . The theorem of this note generalizes this result to a large class of separable x 's.

We assume that x is separable in the sense that there exists a countable subset (t_i) of T such that the closure, with respect to P_x , of the σ -field generated by the $x(t_i)$ contains the σ -field generated by all the $x(t)$. We also assume that the joint distribution of $x(t_i), \dots, x(t_n)$ is given by a density G_n which is almost everywhere positive in R^n with respect to Lebesgue measure. We shall write x_i, y_i, f_i , and g_i for $x(t_i), y(t_i), f(t_i)$, and $g(t_i)$ and S_n for the σ -field generated by x_1, \dots, x_n . It is easily verified that the function D_n on $\Omega \times \Omega$ defined by

$$D_n(f, g) = G_n(f_1 - g_1, \dots, f_n - g_n) / G_n(f_1, \dots, f_n)$$

is, for each fixed g in Ω , a martingale with respect to the fields S_n and the measure P_x . It follows that $D_n(f, g)$ converges almost everywhere (P_x) for each g to a limit $D(f, g)$, that $\int D(f, g) P_x(df) \leq 1$ and that $\int D(f, g) P_x(df) = 1$ if and only if g is in M , in which case $D(f, g) = (dP_{x+g}/dP_x)(f)$ almost everywhere (P_x).

THEOREM. *If x and y are independent processes on T and P and Q are the measures associated with (x, y) and $(x + y, y)$ then $Q < P$ if and only if $P_y(M) = 1$. In this case $(dQ/dP)(f, g) = D(f, g)$.*

PROOF. Suppose $Q < P$. For any $S_n \times S_n$ measurable F

$$\begin{aligned} \int F D_n dP &= \iint F(f_1, \dots, f_n, g_1, \dots, g_n) \\ &\quad \cdot G_n(f_1 - g_1, \dots, f_n - g_n) df_1 \cdots df_n P_y(dg) \end{aligned}$$

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