ORTHOGONALITY IN ANALYSIS OF VARIANCE

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1. Introduction. Although the literature on the subject of analysis of variance is extensive (c.f. Plackett [7]) and goes back a long way, a recent paper by Darroch and Silvey [1] throws a fresh light on the idea of orthogonality which is usually associated with analysis of variance models.

Suppose we have a general linear model $\mathcal{G}:\mathbf{y} = \boldsymbol{\theta} + \boldsymbol{\varepsilon}$, where $\boldsymbol{\varepsilon}$ is a random vector distributed as $N[\mathbf{0}, \sigma^2 I_n]$ and $\boldsymbol{\theta}$, the vector of means, belongs to Ω, a subspace of n-dimensional Euclidean space R^n. Consider a sequence of linear hypotheses $\mathcal{H}_i:\boldsymbol{\theta}$ belongs to ω_i a subspace of Ω $(i = 1, 2, \cdots, K)$. Then from Darroch and Silvey [1] we have the following definition: an experimental design is orthogonal relative to a general linear model $\mathcal{G}$ and linear hypotheses $\mathcal{H}_1, \mathcal{H}_2, \cdots, \mathcal{H}_K$ if the subspaces $\Omega, \omega_1, \omega_2, \cdots, \omega_K$ satisfy the conditions $\omega_i^\perp \cap \Omega \perp \omega_j^\perp \cap \Omega$ for all $i, j, i \neq j$, i.e. if the orthogonal complements of the ω_i with respect to Ω are mutually perpendicular. This definition expresses in general terms the well known orthogonality property of analysis of variance models; namely, that the sums of squares obtained by nesting the hypotheses are stochastically independent and are the same irrespective of the order of the nesting (c.f. Scheffé [11] and Kempthorne [2] p. 49).

In this paper we shall derive necessary and sufficient conditions for a general p-factor analysis of variance model, with unequal observations per cell, to be orthogonal.

2. Matrix conditions for orthogonality. A vector space can be represented in two ways: either as the null space $\mathfrak{N}[A]$ of a matrix A, i.e. $\Omega = \{\boldsymbol{\theta} \mid A\boldsymbol{\theta} = \mathbf{0}\}$ or as the range space $\mathfrak{R}[X]$ of a matrix X, i.e. $\boldsymbol{\theta}$ belongs to Ω if and only if there exists $\boldsymbol{\alpha}$ such that $\boldsymbol{\theta} = X\boldsymbol{\alpha}$. When X is of full rank we have the familiar regression model, while if X is not of full rank and the elements of X are either zero or one, we have the analysis of variance model in which $\boldsymbol{\alpha}$ is not unique and identifiability conditions $H\boldsymbol{\alpha} = \mathbf{0}$ say, are introduced (c.f. Scheffé [11] p. 17). Although the range space representation is the more familiar one, the identifiability conditions can cause theoretical difficulties and so it is often easier to use the null space representation, as shown in Rao [9], Roy and Roy [10] and in the theory below. We shall require the following lemma.

Lemma. *If*

$$\Omega = \{\boldsymbol{\theta} \mid A\boldsymbol{\theta} = \mathbf{0}\}, \qquad \omega_i = \{\boldsymbol{\theta} \mid A\boldsymbol{\theta} = \mathbf{0}, A_i\boldsymbol{\theta} = \mathbf{0}\}$$

where the rows of the matrix $[A' : A_i']'$ *are linearly independent,* $(i = 1, 2, \cdots, K)$, *and* $AA_i' = 0$ *for* $i = 1, 2, \cdots, K$, *then* $\omega_i^\perp \cap \Omega \perp \omega_j^\perp \cap \Omega$ *if and only if* $A_iA_j' = 0$.

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