ON THE LIFTING PROPERTY (V)[1]

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1. Let $(X, \mathcal{B}, \mu)$ be a measure space (i.e. X is a set, $\mathcal{B}$ a σ-algebra of subsets of X, μ a positive countably additive measure on $\mathcal{B}$). Let $\mathcal{B}_0 = \{B \in \mathcal{B} \mid \mu(B) < \infty\}$ and $\mathfrak{N} = \{A \in \mathcal{B} \mid \mu(A) = 0\}$. For $A \in \mathcal{B}$, $B \in \mathcal{B}$ we write $A \equiv B$ if $A \Delta B = (A - B) \cup (B - A) \in \mathfrak{N}$; this is an equivalence relation in $\mathcal{B}$. We shall denote by $B \to \tilde{B}$ the canonical mapping of $\mathcal{B}$ onto the quotient σ-algebra $\mathcal{B}/\mathfrak{N}$. *Throughout this paper we shall assume that the measure space* $(X, \mathcal{B}, \mu)$ *satisfies the following conditions:*

(a) The measure space $(X, \mathcal{B}, \mu)$ is complete (i.e., the relations $A \in \mathfrak{N}$ and $B \subset A$ imply $B \in \mathfrak{N}$);

(b) A set $E \subset X$ belongs to $\mathcal{B}$ if and only if $E \cap B \in \mathcal{B}$ for every $B \in \mathcal{B}_0$;

(c) For every $E \in \mathcal{B}$, $\mu(E) = \sup \{\mu(B) \mid Bc \subset A, B \in \mathcal{B}_0\}$;

(d) The quotient σ-algebra $\mathcal{B}/\mathfrak{N}$ is a complete lattice.

The measure space $(X, \mathcal{B}, \mu)$ is then a localizable measure space in Segal's sense (see [21] and [13]).

Note that the above setting includes as a particular case $(X, \mathcal{B}, \mu)$ a complete totally σ-finite measure space. Also, if X is a locally compact space with a given positive Radon measure, the conditions (a)–(d) are satisfied if we take for $\mathcal{B}$ the σ-algebra of all sets measurable with respect to that Radon measure and for μ the essential measure (see [1]).

In what follows we shall denote by M_R^∞ the algebra of all bounded real-valued measurable functions defined on X. For $f \in M_R^\infty$, $g \in M_R^\infty$ we write $f \equiv g$ if f and g coincide almost everywhere; this defines an equivalence relation in M_R^∞. As usual, we denote by L_R^∞ the quotient space of M_R^∞ under this equivalence relation, and by $f \to \tilde{f}$ the canonical mapping of M_R^∞ onto L_R^∞. Endowed with the essential supremum norm, L_R^∞ is a cummutative Banach algebra.

Let now $T: f \to T_f$ be a mapping of M_R^∞ into M_R^∞ and consider the following axioms:

(I) $T_f \equiv f$;

(II) $f \equiv g$ implies $T_f = T_g$;

(III) $T_1 = 1$;

(IV) $f \geqq 0$ implies $T_f \geqq 0$;

(V) $T_{\alpha f + \beta g} = \alpha T_f + \beta T_g$;

(VI) $T_{fg} = T_f T_g$.

Let us recall that a mapping $T: f \to T_f$ of M_R^∞ into M_R^∞ satisfying (I)–(VI) is called a *lifting of* M_R^∞; a mapping $T: f \to T_f$ of M_R^∞ into M_R^∞ satisfying (I)-(V) is called a *linear lifting of* M_R^∞ (see [10]).

Received 21 December 1964

[1] Research supported by the U. S. Army Research Office (Durham), under contract DA-31-124-ARO-D-288.