ANOTHER CHARACTERISTIC PROPERTY OF THE CAUCHY DISTRIBUTION

By M. V. Menon[1]

IBM Research Labs., San Jose, Calif.

1. Main result. The purpose of this paper is to prove the theorem and corollary stated in this section. The corollary answers the question raised in Section 3 of [4] to which we also refer the reader for further motivation.

Theorem. *Let X be a symmetric* r.v., *and X_i, $i = 1, 2, \cdots$,* r.v.'s *independently and identically distributed as X. The two conditions:* (i) *for any real number c, and positive integer n, there exist real numbers $A = A(n, c)$ and $B = B(n, c)$ for which $\sum_1^n 1/(X_i + c)$ is distributed as $A/(X + B)$, and* (ii) *for some $c \neq 0$ the symmetric* r.v. $1/(X_1 + c) + 1/(X_2 - c)$ *is distributed as $A(c)/X$, for some number $A(c)$, are necessary and sufficient for X to have a Cauchy distribution.*

Corollary. *Let X_i, $i = 1, 2, \cdots$ be* r.v.'s *independently and identically distributed as a* r.v. X. *The necessary and sufficient condition that, for any real numbers $a_i \neq 0$, b_i, $i = 1, 2, \cdots$ and any positive integer n, there exist real numbers A and B for which $\sum_1^n 1/(a_iX_i + b_i)$ has the same distribution as $A/(X + B)$, is that X have the Cauchy distribution.*

2. Notation. We set down some of the notation used which will conform to that of [4] as far as possible. $\xi_1, \xi_2, \cdots$, denote r.v.'s independently and identically distributed as any given r.v. ξ. When two r.v.'s ξ and η are set equal to each other, $\xi = \eta$, we mean only that they have the same distribution, and *not* that they are equal with probability one. The symbols $\rightarrow_p$ and $\rightarrow_w$ stand for convergence in probability and weak convergence respectively. Slightly modifying the usual meaning of the symbol $\sim$, we write $f_n \sim g_n$, if f_n/g_n converges to a positive constant as n tends to infinity. The pdf (probability density function) and ch.f. (characteristic function) of $1/X$ will be denoted by $f(x)$ and $\phi(t)$ resp., and the pdf of X by $g(x)$. (That the pdf's mentioned exist will follow in the course of the proof.) Finally, by $\mathcal{L}(\xi)$ is meant the law of any r.v. ξ.

N.B. Only non-degenerate r.v.'s are considered in this paper.

3. Proofs. We first prove that the conditions stated in the theorem are sufficient. Therefore, let

$$\sum_1^n 1/(X_i + c) = A(n, c)/[X + B(n, c)] \tag{3.1}$$

where $A(n, c)$ is assumed, without any loss of generality, to be positive, and

$$1/(X_1 + c) + 1/(X_2 - c) = A(c)/X \tag{3.2}$$

where $c \neq 0$.

Received 12 December 1964; revised 1 April 1965.

[1] Now at Math. Res. Centre, Univ. of Wisconsin.