A NOTE ON MINIMUM DISCRIMINATION INFORMATION[1]

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This note contains a simple proof of the minimum discrimination information theorem in Kullback (1959), pp. 36–39 and an affirmative answer to a suggestion in a personal communication from Dr. I. J. Good that the theorem could be applied even to random elements of a Banach space.

Let X be a space of points x, S a σ-field of sets of X, and P_2 a probability measure on S. Let $T(x)$ be a real valued S-measurable function such that

$$M_2 = \int_X \exp(T(x))\, dP_2 < \infty \tag{1}$$

and let the probability measure P^* be defined by

$$P^*(A) = \int_A (\exp(T(x))/M_2)\, dP_2, \qquad \text{for } A \varepsilon S. \tag{2}$$

Suppose that $T(x)$ is P^*-integrable, and let

$$\theta = \int_X T(x)\, dP^*. \tag{3}$$

Now let P_1 be an arbitrary probability measure on S. If $P_1 \ll P_2$ define

$$I(P_1, P_2) = \int_X [\log(dP_1/dP_2)]\, dP_1, \tag{4}$$

otherwise define $I(P_1, P_2) = \infty$. It is clear that $I(P^*, P_2)$ is finite; in fact, from (2) and (3),

$$I(P^*, P_2) = \theta - \log M_2. \tag{5}$$

Theorem. *If P_1 is a probability measure on S such that T is P_1-integrable and*

$$\int_X T(x)\, dP_1 = \theta, \tag{6}$$

then

$$I(P_1, P_2) \geqq I(P^*, P_2) = \theta - \log M_2 \tag{7}$$

with equality if and only if $P_1 = P^*$ on S.

Proof. If $I(P_1, P_2) = \infty$ there is nothing to prove. Suppose then that $I(P_1, P_2) < \infty$. In this case $P_1 \ll P_2$, and we write $f(x) = dP_1/dP_2$. Then

$$I(P_1, P_2) = \int_X f(x) \log f(x)\, dP_2 \tag{8}$$

and

$$I(P^*, P_2) = \int_X f^*(x) \log f^*(x)\, dP_2 \tag{9}$$

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