PERMUTATIONS WITHOUT RISING OR FALLING w-SEQUENCES

BY MORTON ABRAMSON AND W. O. J. MOSER

McGill University

0. Introduction. A permutation of degree $n > 1$ is said to contain the sequence $ijk \cdots st$ if, in the permutation, i immediately precedes j, j immediately precedes $k, \cdots$, and s immediately precedes t. The rising w-sequences ($w \geqq 2$) are those in the left column in Table I; the falling w-sequences are in the right column.

The enumeration of permutations without rising 2-sequences is given by Whitworth [8], p. 102. Permutations without rising or falling 2-sequences have been treated by Kaplansky [2] in the form of what he calls the n king problem: in how many ways can n kings be placed on an $n \times n$ board, no two in a row or column and no two attacking each other? (See also [1], [3], [4], [6], [9].) Riordan [5] enumerated permutations without rising 3-sequences. In Section 1 we obtain expressions for the number of permutations containing exactly $r \geqq 0$ rising and/or falling w-sequences. In Section 2 we obtain corresponding results in the "circular" case, where the integers 1 and n are considered adjacent.

1. Straight line case. Call a k-choice

$$x_1 < x_2 < \cdots < x_k \tag{1}$$

from $\{1, 2, \cdots, n\}$ a $(n\colon k \mid a, b, c, \cdots)$-choice if

$$a = \textstyle\sum_{x_i - x_{i-1} > 1} 1, \qquad b = \sum_{x_i - x_{i-1} > 2} 1, \qquad c = \sum_{x_i - x_{i-1} > 3} 1, \cdots. \tag{2}$$

Clearly $a \geqq b \geqq c \geqq \cdots$, and any $(n\colon k \mid a, b, \cdots, p, q)$-choice is also a $(n\colon k \mid a, b, \cdots, p)$-choice (but not in general conversely). For example, for $n \geqq 21$,

$$2, 3, 5, 9, 10, 11, 13, 17, 18, 19, 20, 21 \tag{3}$$

is a $(n\colon 12 \mid 4, 2, 2, 0, \cdots)$-choice. Let $((n\colon k \mid a, b, c, \cdots, p, q))$ denote the number of $(n\colon k \mid a, b, c, \cdots, p, q)$-choices.

As usual we take

$$\binom{n}{r} = n!/r!(n-r)! \quad \text{when} \quad 0 \leqq r \leqq n,$$
$$= 0 \qquad \text{otherwise.}$$

THEOREM 1.

$$((n\colon k \mid a, b, c, \cdots, p, q)) = \binom{k-1}{a}\binom{a}{b}\binom{b}{c} \cdots \binom{p}{q}\binom{n-k-a-b-\cdots-p+1}{q+1}, \quad 1 \leqq k \leqq n. \tag{4}$$

We require the well-known [7], p. 92,

LEMMA. *The number of ways of putting n like objects into m differents cells is* $\binom{n+m-1}{m-1} = \binom{n+m-1}{n}$. *When no cell is empty the number of ways is* $\binom{n-1}{m-1}$.

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