The Annals of Mathematical Statistics
1968, Vol. 39, No. 4, 1354–1357

CONSTRUCTION OF JOINT PROBABILITY DISTRIBUTIONS

By L. F. Kemp, Jr.

Polytechnic Institute of Brooklyn

1. Introduction. In (1), (2), (4), (5), and (6), there are constructions of joint probability distributions having given marginal distributions. By a generalization of an exercise from Halmos' *Measure Theory*, we construct a class of doubly stochastic measures as the Lebesgue integral of special simple and elementary functions whose values must be positive and satisfy a dependent system of linear equations. Employing this construction, we get an additional method for generating joint distributions with given marginal distributions.

Let $F_1(x)$ and $F_2(y)$ be the distribution functions of two random variables. Frechet proved that the family of joint distributions having $F_1(x)$ and $F_2(y)$ as marginal distributions collapses to $F_1(x)F_2(y)$ if and only if either $F_1(x)$ or $F_2(y)$ is a unit step function. We rephrase his result in terms of abstract probability measures and with the aid of the above construction of doubly stochastic measures, we show his result is equivalent to the statement that Cartesian product measure is an extreme point of the set of doubly stochastic measures.

2. The construction. (X, S, μ) and (Y, T, ν) are two abstract probability triples. $(X \times Y, S \times T)$ is the Cartesian cross product measure space of the measure spaces (X, S) and (Y, T). λ is called a *doubly stochastic* measure on $(X \times Y, S \times T)$ if

$$\lambda(A \times Y) = \mu(A), \qquad \text{for all} \quad A \quad \text{in} \quad S;$$

$$\lambda(X \times B) = \nu(B), \qquad \text{for all} \quad B \quad \text{in} \quad T.$$

Cartesian product measure $\mu \times \nu$ is a doubly stochastic measure.

Theorem 1. *Let $\{A_i\}$ and $\{B_j\}$ be finite or countably infinite measurable partitions of X and Y, respectively, then the set function*

$$\lambda(E) = \int_E \sum_{i,j} \alpha_{ij} K_{A_i \times B_j}\, d(\mu \times \nu), \qquad \text{for} \quad E \quad \text{in} \quad S \times T$$

is a doubly stochastic measure, if and only if,

(1) $\alpha_{ij} \geqq 0$, all i, j;
(2) $\sum_i \alpha_{ij}\mu(A_i) = 1$, for every j;
(3) $\sum_j \alpha_{ij}\nu(B_j) = 1$, for every i;

where $K_{A_i \times B_j}$ is the characteristic function of the rectangle $A_i \times B_j$.

Proof. Before we begin the proof, let us assume without loss of generality that $\mu(A_i)\nu(B_j) > 0$ all i, j since $\lambda \ll \mu \times \nu$.
Sufficiency: If $\alpha(x, y) = \sum_{i,j} \alpha_{ij} K_{A_i \times B_j}(x, y)$, then $\alpha(x, y)$ is a positive measurable elementary function so that λ is a measure. If $\alpha_n(x, y) =$

Received 21 September 1967.