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EFFICIENT DIFFERENCE EQUATION ESTIMATORS IN EXPONENTIAL REGRESSION

By C. A. McGilchrist[1]

University of New South Wales

1. Introduction. A multiple exponential regression curve is given by

$$\epsilon(Y_x) = \eta_x = \alpha - \textstyle\sum_{i=1}^{k} \beta_i \rho_i^{\,x}, \qquad 0 < \rho_1 < \cdots < \rho_k < 1, \tag{1}$$

where Y_x is the observation at x. We consider here the estimation of the ρ_i when the observations are independent and normally distributed with constant variance, σ^2, and are equally spaced at x values denoted by $x = 0, 1, 2, \cdots, n - 1$.

For single exponential regression ($k = 1$) a technique used by Hartley [3] and Patterson [6] is to replace the regression curve by a difference equation which generates it and to estimate the parameters of the difference equation. Using this technique Lipton and McGilchrist [5] have obtained a class of estimators for the ρ_i of the multiple exponential model. Denoting an estimator of ρ_i by r_i, this class is given by the solution of the equations,

$$\begin{aligned}[(-1)^k\hat\theta_k\mathbf{y}_0' + (-1)^{k-1}\hat\theta_{k-1}\mathbf{y}_1' + \cdots + \mathbf{y}_k']D[(-1)^k(\hat\theta_k^{\,j}b_j + \hat\theta_k)\mathbf{y}_0 \\ + (-1)^{k-1}(\hat\theta_{k-1}^{j}b_j + \hat\theta_{k-1})\mathbf{y}_1 + \cdots + \mathbf{y}_k] = 0, \quad j = 1, 2, \cdots k,\end{aligned} \tag{2}$$

where

$$\begin{aligned}\mathbf{y}_p &= \{Y_p, Y_{p+1}, \cdots, Y_{n+p-k-1}\}, \qquad p = 0, 1, 2, \cdots, k,\\ \hat\theta_p &= p\text{th order symmetric function in } r_1, r_2, \cdots, r_k,\\ \hat\theta_p^{\,j} &= \partial\hat\theta_p/\partial r_j .\end{aligned}$$

The above equations correspond to those given on p. 507 of [5]. The b_j are constants and D is a square matrix of order $n - k$, and the b_j and D are selected to satisfy suitable criteria. Except for the case of single exponential regression (already studied by Patterson [6]), Lipton and McGilchrist found the usual criteria of zero bias and minimum variance too difficult to apply to (2) in order to select the b_j and D, and were unable to proceed. In this paper alternative criteria are considered in Section 2 and these found much easier to apply in Section 4.

2. Estimating equations and criteria. The criteria now described are similar in principle to those suggested by Barnard and reported by Durbin [2]. Representing estimating equations (2) by

$$T_j(Y, r) = 0, \qquad j = 1, 2, \cdots, k, \tag{3}$$

we consider the equivalent estimating functions, $T_j(Y, \rho)$, which are obtained

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