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SOME RESULTS ON THE COMPLETE AND ALMOST SURE CONVERGENCE OF LINEAR COMBINATIONS OF INDEPENDENT RANDOM VARIABLES AND MARTINGALE DIFFERENCES[1]

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1. Introduction. Let $(\Omega, \mathfrak{F}, P)$ be a probability space with $(\mathfrak{F}_{k;k\geqq 1})$ an increasing sequence of σ-fields such that $\mathfrak{F}_k \subset \mathfrak{F}$. Let $(D_k, \mathfrak{F}_{k,k\geqq 1})$ be a martingale difference sequence; i.e., each D_k is $\mathfrak{F}_k$ measurable and $E(D_k \mid \mathfrak{F}_{k-1}) = 0$ a.s. for all $k \geqq 2$. Let a_{nk} be a matrix of real numbers,

$$A_n = \sum_{k=1}^{\infty} a_{nk}^2, \qquad T_{nm} = \sum_{k=1}^{m} a_{nk} D_k \quad \text{and} \quad T_n \text{ be the a.s. limit}$$

of T_{nm} as $m \to \infty$ whenever this limit exists. T_n is said to converge completely to zero in the sense of Hsu and Robbins [8] if $\sum_{n=1}^{\infty} P[|T_n| > \epsilon] < \infty$ for all $\epsilon > 0$. It should be noted that T_n converging completely to zero implies that T_n converges a.s. to zero and that the two types of convergence are equivalent if the T_n's form a sequence of independent random variables. The purpose of this paper is to present various sets of conditions for the complete or a.s. convergence of T_n to zero.

Sections 3 and 4 deal with the special case where the $(D_k, k \geqq 1)$ are independent random variables, Section 3 treating the identically distributed case and Section 4 treating the non-identically distributed case. The results given in these two sections extend or improve results given by Hsu and Robbins [8], Erdös [4], Pruitt [11], and Chow [1]. The double truncation method of proof developed by Erdös [4] and improved by other authors ([1], [5], and [11] for example) is fundamental. The work of Franck and Hanson [5] is closely related to that presented here. The main results are given by Theorems 1 and 3 with more specific applications given by Corollaries 1–3. Theorem 2 is of special interest since it shows that the double truncation method of Erdös used in [4] to obtain sharp results about complete convergence can sometimes be modified to obtain sharp results about almost sure convergence.

According to Chow [1], a random variable D is generalized Gaussian if there exists an $\alpha \geqq 0$ such that for every real t, $E \exp(tD) \leqq \exp(t^2\alpha^2/2)$. The minimum of these numbers α is denoted by $\tau(D)$. Special cases of generalized Gaussian random variables include normal and bounded random variables each with mean zero. (See [1], p. 1482.) In Section 5 we extend to the martingale case a result of Chow ([1], p. 1483) concerning the complete convergence of T_n to zero when the $(D_k, k \geqq 1)$ are independent and generalized Gaussian with $\tau^2(D_k) \leqq 2$.

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