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SOME INTEGRAL TRANSFORMS OF CHARACTERISTIC FUNCTIONS[1]

BY G. ANDERSEN AND T. KAWATA

The Catholic University of America

1. Introduction. We shall deal with the basic convergence problem of sequences of probability distribution functions in relation to the celebrated Lévy continuity theorem. One of the simplest ways of proving this theorem was adopted by M. Loève [9]; it employs the integral characteristic function

$$f^I(x) = \int_0^x f(t)\,dt = \int_{-\infty}^{+\infty} (e^{ixu} - 1)/iu\, dF(u), \tag{1.1}$$

where $f(t)$ is the characteristic function of the distribution function $F(u)$. This method was also adopted in [7].

A standard form of the Lévy continuity theorem requires the continuity at the origin of the limiting function, $f(u)$, of a sequence of characteristic functions. L. Schmetterer [10] has shown that the continuity of $f(u)$ can be replaced by $(C, 1)$ summability of the Fourier series of $f(u)$ to one at the origin. So he considered the Fejér integral of a characteristic function $f(u)$:

$$L(\alpha) = \pi^{-1} \int_{-\infty}^{+\infty} \sin^2 \alpha t(\alpha t^2)^{-1} f(t)\,dt, \qquad (\alpha > 0). \tag{1.2}$$

Further, it was noticed by one of the authors [5], that the transform

$$L_0(x, \alpha) = \pi^{-1} \int_{-\infty}^{+\infty} \sin^2 \alpha(t - x)(\alpha(t - x)^2)^{-1} f(t)\,dt, \tag{1.3}$$

$\alpha > 0, -\infty < x < +\infty$, plays a role similar to that of $f^I(x)$ in the convergence problem of sequences of distribution functions.

Here, in place of (1.3), we will take the Fourier transform of the c.f. after multiplication by the Fejér kernel:

$$L(x, \alpha) = \pi^{-1} \int_{-\infty}^{+\infty} \sin^2 \alpha t(\alpha t^2)^{-1} f(t) e^{-itx}\,dt, \tag{1.4}$$

$\alpha > 0, -\infty < x < +\infty$. The use of this functional allows for simpler arguments, concerning the convergence problem, than (1.2) and, in some sense, even $f^I(x)$. This is based on the well-known and easily shown fact that

$$L(x, \alpha) = (2\alpha)^{-1} \int_0^{2\alpha} (F(x + u) - F(x - u))\,du, \tag{1.5}$$

where $F(x)$ is the df corresponding to the cf $f(t)$ in (1.4). In Section 2 we will illustrate the different roles played by (1.4) and (1.5).

As pointed out by L. Schmetterer [10], the Fejér kernel may be replaced by other summability kernels. Even so, it is worthwhile to approach this problem also with the Poisson integral of a characteristic function

$$f(x, y) = \int_{-\infty}^{+\infty} P(t - x, y) f(t)\,dt, \tag{1.6}$$

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