The Annals of Mathematical Statistics
1969, Vol. 40, No. 2, 505–508

A COMPARISON TEST FOR MARTINGALE INEQUALITIES

By Burgess Davis

Rutgers—The State University

1. Introduction. If $f = (f_1, f_2, \cdots)$ is a sequence of real valued functions on a probability space we define its difference sequence $d = (d_1, d_2, \cdots)$ by $d_1 = f_1$, $d_i = f_i - f_{i-1}$, $i > 1$, and use the following notation: $f_n^* = \max(|f_1|, \cdots, |f_n|)$, $f^* = \sup_n f_n^*$, $S_n(f) = [\sum_{i=1}^n d_i^2]^{\frac{1}{2}}$, $S(f) = S_\infty(f) = \sup_n S_n(f)$, and $\|f\|_p = \sup_n \|f_n\|_p$, where $\|f_k\|_p$ is the L^p norm of f_k. $\|f\|_p$ will be called the L^p norm of the sequence f and f will be said to be L^p bounded if it has finite L^p norm.

In [2] Burkholder derives a number of martingale inequalities from Theorem 6 of that paper, which states: There is a real number M such that if f and g are martingales relative to the same sequence of σ-fields and $S_n(g) \leqq S_n(f)$, $n \geqq 1$, then $\lambda P(g^* > \lambda) \leqq M \|f\|_1$, $\lambda > 0$.

The proof of this result is based on a widely applicable method, which yields, however, no information about the size of M. In [6] Gundy gives proofs capable of providing numerical bounds for M for several of the inequalities established in [2]. However, only a special case of Theorem 6 is obtained, that in which g is a transform of f under a uniformly bounded multiplier sequence. Here a proof providing numerical bounds for M is given for a strengthened version of Theorem 6 and an additional inequality is obtained for g^* if the f of Theorem 6 is uniformly integrable.

In the final section several existing results about the convergence of L^1 bounded martingales f are shown to follow easily from information concerning $S(f)$.

2. A comparison test for martingale inequalities. If $f = (f_1, f_2, \cdots)$ is a martingale with difference sequence d then

$$E(f_n^2) = E((\textstyle\sum_{i=1}^n d_i)^2) = E(\sum_{i=1}^n d_i^2) = E((S_n(f))^2).$$

Since $E(f_n^{*2}) \leqq 4E(f_n^2)$ by an inequality due to Doob ([4], page 317) we have upon taking limits the basic relation:

$$E(S(f)^2) \leqq E(f^{*2}) \leqq 4E(S(f)^2). \tag{1}$$

We will make use of the result proved in [2] and [4], if f is a martingale

$$\lambda P(S(f) > \lambda) \leqq 22 \|f\|_1, \qquad \lambda > 0, \tag{2}$$

(Burkholder proves there is a real number M for which $\lambda P(S(f) > \lambda) \leqq M \|f\|_1$, and Gundy's method gives numerical bounds for M). In particular (1) will be used to translate the information (2) gives us about $S(f)$ into information about f^*. No effort is made to minimize the constants involved.

Received 11 July 1968.