The Annals of Mathematical Statistics
1969, Vol. 40, No. 2, 462–466

ON THE LEAST SQUARES ESTIMATION OF NON-LINEAR RELATIONS

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1. Introduction. Consider a non-linear relation

$$y = f(x_1, \cdots, x_m; \alpha_1, \cdots, \alpha_p) \tag{1.1}$$

among the real variables $x_1, \cdots, x_m$ and y, where f is a known function and $\alpha_1, \cdots, \alpha_p$ are unknown parameters. The problem of estimating these parameters by least squares methods has been considered recently by Hartley and Booker [1], assuming that the variables x_j are not subject to error and that the variable y is observed with an error which is normally distributed. In this paper, which is only a complement to a previous paper on linear relations [2], these assumptions will be dropped, but it will be assumed instead that replicated observations are available.

2. Notation and model. Suppose that, in order to estimate the non-linear relation (1.1), we have performed an experiment with n replications, which may possibly have an incomplete block design, and suppose that, from the usual statistical analysis of the data, we have obtained $k(m + 1)$ estimators x_{ijn}, y_{in} $(i = 1, \cdots, k; j = 1, \cdots, m)$ converging in probability, when n tends to infinity, to values x_{ij}, y_i which satisfy the non-linear relation (1.1). Using the vector space approach of [2], we shall consider an auxiliary p-dimensional Euclidian space $\mathfrak{U}$ with an orthonormal basis $\mathbf{u}_1, \cdots, \mathbf{u}_p$ and an m-dimensional Euclidian space $\mathfrak{V}$ with an orthonormal basis $\mathbf{v}_1, \cdots, \mathbf{v}_m$ and we shall define on them the linear functionals A, X_{in}, X_i by $A\mathbf{u}_h = \alpha_h$ $(h = 1, \cdots, p)$; $X_{in}\mathbf{v}_j = x_{ijn}$ and $X_i\mathbf{v}_j = x_{ij}$. Then we have

$$y_i = f(X_i, A). \tag{2.1}$$

We shall assume that f is a continuously differentiable function. The value of the differential of f at an arbitrary point $(X, A^*; \Delta X, \Delta A)$ will be denoted by $\Delta X f_X(X, A^*) + \Delta A f_A(X, A^*)$, where $f_A(X, A^*) \in \mathfrak{U}$ and $f_X(X, A^*) \in \mathfrak{V}$. We shall write simply

$$\mathbf{f}_i = f_A(X_i, A), \qquad \mathbf{f}_{Xi} = f_X(X_i, A), \tag{2.2}$$

and we shall assume that the vectors $\mathbf{f}_i$ do not lie on any proper subspace of $\mathfrak{U}$. We assume, in addition, that the joint distribution of the $2k$ random variables $n^{\frac{1}{2}}\Delta y_{in}$ (where $\Delta y_{in} = y_{in} - y_i$) and $n^{\frac{1}{2}}\Delta X_{in}$ (where $\Delta X_{in} = X_{in} - X_i$) con-

Received 2 May 1967; revised 29 July 1968.

[1] Now visiting at the University of Rochester. This work has been made with a special support from the Universidad de la República, Montevideo.