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INFINITELY DIVISIBLE RENEWAL DISTRIBUTIONS

BY F. W. STEUTEL

Technische Hogeschool Twente, Enschede, Netherlands

1. Introduction. It is well-known that many waiting-time processes yield infinitely divisible (inf div) waiting-time distributions, independent of the divisibility properties of the arrival and service time distributions. One might expect that renewal processes also would often yield inf div renewal distributions. It turns out however that even when the original distribution (with characteristic function (cf) $\phi(t)$) is inf div then the renewal distribution (with cf $\{\phi(t) - 1\}(i\mu t)^{-1}$) need not be inf div On the other hand a distribution, which is not inf div may give rise to an inf div. renewal distribution. In this paper conditions, in terms of the original distribution function, are derived for the renewal distribution to be inf div. In relation with this the infinite divisibility of some waiting-time distributions is considered. It is shown that not all waiting-time distributions are inf div.

2. Preliminaries. We will be concerned with distributions having a probability density function (pdf) of the form

$$g(x) = \mu^{-1}(1 - F(x)), \tag{1}$$

where $F(x)$ is the distribution function of a non-negative random variable with mean $\mu > 0$. If F is a distribution function, then by F^{*k} we denote the kth convolution of F with itself. The derivative of F^{*k} will be denoted by f^{*k}. The Laplace-Stieltjes transform (LT) of distribution functions F, G, $\cdots$ are denoted by $\tilde{F}(\tau)$, $\tilde{G}(\tau)$, $\cdots$, their Fourier-Stieltjes transforms (cf's) by $\phi(t)$, $\gamma(t)$, $\cdots$. In the waiting-time examples we use Wishart's notation. We will have to consider the function $L(x)$ defined by

$$L(x) = \sum_{k=1}^{\infty} k^{-1}F^{*k}(x), \tag{2}$$

which has been studied by Smith [3].

LEMMA 1. *For all* $\tau > 0$

$$\sum_{k=1}^{\infty} k^{-1} \int_0^{\infty} e^{-\tau x}x\, dF^{*k}(x) = \int_0^{\infty} e^{-\tau x}x\, dL(x).$$

PROOF. By Fubini's theorem (partial integration) we have

$$\int_0^{\infty} e^{-\tau x}x\, dF^{*k}(x) = \int_0^{\infty} e^{-\tau x}(\tau x - 1)F^{*k}(x)\, dx. \tag{3}$$

Summation of (3) and the use of Fubini's theorem to invert the order of summation and integration yields

$$\sum_{k=1}^{\infty} k^{-1} \int_0^{\infty} e^{-\tau x}x\, dF^{*k}(x) = \int_0^{\infty} e^{-\tau x}(\tau x - 1)L(x)\, dx.$$

Using partial integration once more we obtain the required result.

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