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# MONOTONICITY OF THE VARIANCE UNDER TRUNCATION AND VARIATIONS OF JENSEN'S INEQUALITY[1]

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Let $X$ be a random variable and $Y = XI_{[X<b]}$ for $-\infty < b < \infty$. Then it is easy to construct examples for which $\sigma^2X < \sigma^2Y$; e.g. take $X = b \pm \epsilon$ with probability $\frac{1}{2}$ where $\epsilon \ll b$. However if $Y = \min(X, b)$ then we always have $\sigma^2X \geqq \sigma^2Y$. The essential difference is that we are now replacing any value $X > b$ by $b$ instead of 0. Similarly if $Y = \max(a, X)$ or $\max(a, \min(X, b))$ then $\sigma^2X \geqq \sigma^2Y$. The main result in this note is Theorem 2 and Corollary 4 which contains a proof of a conditional version of the above result which was used in [2]. We will prove the above facts and some of their generalizations which provide intermediate terms in Jensen's inequality; see Corollary 1. The results and the methods of proof given below are actually special cases or slight modifications of more general inequalities involving duals of cones of generalized convex functions. See for example Karlin and Novikoff [3], Karlin and Studden [4], Ziegler [5], Barlow, Marshall and Proschan [1] and references therein. The methods used below are quite elementary and produce the desired results without recourse to dual cones.

THEOREM 1. *Let $h$ and $g$ be two Baire functions on the real line and $X$ a random variable such that* (a) *$g$ is nondecreasing,* (b) *$Eg(X)$ and $Eh(X)$ exist and $Eg(X) = Eh(X)$,* (c) *the function $h - g$ has one sign change from negative to positive,* i.e., *there exists $t_0$ such that $(h(t) - g(t))(t - t_0) \geqq 0$ for every $t$. If $E\varphi(h(X))$ and $E\varphi(g(X))$ exist, then*

$$E\varphi(h(X)) \geqq E\varphi(g(X)) \tag{1}$$

*for all continuous, convex $\varphi$ on $(-\infty, \infty)$.*

PROOF. By the convexity of $\varphi$,

$$\varphi(h(t)) - \varphi(g(t)) \geqq \varphi'(g(t))(h(t) - g(t)), \tag{2}$$

where $\varphi'$ denotes the right (or left) derivative of $\varphi$. If $t > t_0$, then $h(t) - g(t) \geqq 0$ and $\varphi'(g(t)) \geqq \varphi'(g(t_0))$, so that

$$\varphi(h(t)) - \varphi(g(t)) \geqq \varphi'(g(t_0))(h(t) - g(t)). \tag{3}$$

The above equation also holds for $t < t_0$, since $h(t) - g(t) \leqq 0$ and $\varphi'(g(t)) \leqq \varphi'(g(t_0))$. Therefore (3) holds for all $t$. (1) follows immediately from (3).

COROLLARY 1. *Suppose that $f$ is nondecreasing, $\varphi$ is continuous, convex and for a random variable $X$, $EX$ and $Ef(X)$ exist, and $t - f(t) + Ef(X) - EX$ has one*

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