The Annals of Mathematical Statistics
1969, Vol. 40, No. 3, 836–843

ON THE EXACT DISTRIBUTIONS OF VOTAW'S CRITERIA FOR TESTING COMPOUND SYMMETRY OF A COVARIANCE MATRIX

BY P. C. CONSUL

University of Calgary

1. Introduction. Let x_{ij}, for $i = 1, 2, 3, \cdots, n$, be n independent observations on $p + q$ stochastic variables X_j, where $j = 1, 2, 3, \cdots, (p + q)$, which are distributed normally. Also, let

$$(1.1)\qquad \bar{x}_j = n^{-1} \textstyle\sum_{i=1}^{n} x_{ij}, \qquad s_{jj'} = n^{-1} \textstyle\sum_{i=1}^{n} (x_{ij} - \bar{x}_j)(x_{ij'} - \bar{x}_{j'}).$$

Also, let $S = ((s_{ij'}))$ be the sum of products (SP) matrix for X's and

$$(1.2)\qquad \begin{aligned} S_{aa} &= p^{-1} \textstyle\sum_{j=1}^{p} S_{jj}, & S_{aa'} &= 2(p^2 - p)^{-1} \textstyle\sum_{j>j'=1}^{p} S_{jj'}, \\ S_{bb} &= q^{-1} \textstyle\sum_{j=p+1}^{p+q} S_{jj}, & S_{bb'} &= 2(q^2 - q)^{-1} \textstyle\sum_{j>j'=p+1}^{p+q} S_{jj'}, \\ S_{ab} &= (pq)^{-1} \textstyle\sum_{j=1}^{p} \sum_{j'=p+1}^{p+q} S_{jj'}. \end{aligned}$$

To test the hypothesis H that the covariance matrix is of the bipolar form

$$(1.3)\qquad \begin{bmatrix} \Sigma_1 & \Sigma_2 \\ \Sigma_2' & \Sigma_3 \end{bmatrix}$$

where Σ_1 is a $p \times p$ matrix with diagonal elements equal to σ_{aa} and other elements to $\sigma_{aa'}$, Σ_2 is a $p \times q$ matrix with all elements equal to σ_{ab} and Σ_3 is a $q \times q$ matrix with diagonal elements σ_{bb} and other elements $\sigma_{bb'}$, the likelihood ratio statistic can be defined by

$$(1.4)\qquad L = |S|/\{[(S_{aa} + (p - 1)S_{aa'})(S_{bb} + (q - 1)S_{bb'}) - pqS_{ab}^{2}\} \cdot (S_{aa} - S_{aa'})^{p-1}(S_{bb} - S_{bb'})^{q-1}].$$

Votaw (1948) used Wilks' (1934) moment generating operator and derived the expected value $E(L^t \mid H)$ when the hypothesis H was true. By orthogonal transformation and by integrating over the range of different variates Roy (1951) proved that the expected value $E(L^t \mid H)$ i.e. the tth moment can be expressed in the form

$$(1.5)\qquad \begin{aligned} E(L^t \mid H) = \{&(p - 1)^{p-1}(q - 1)^{q-1}\}^t \\ &\cdot \left[\frac{\Gamma\{\frac{1}{2}(q - 1)(n - 1)\}\Gamma\{\frac{1}{2}(p - 1)(n - 1)\}}{\Gamma\{(p - 1)(t + \frac{1}{2}(n - 1))\}\Gamma\{(q - 1)(t + \frac{1}{2}(n - 1))\}}\right] \\ &\cdot \textstyle\prod_{r=0}^{p+q-3} \Gamma\{t + \frac{1}{2}(n - 3) - \frac{1}{2}r\}[\Gamma\{\frac{1}{2}(n - 3) - \frac{1}{2}r\}]^{-1} \end{aligned}$$

and obtained the distribution of L in the form of an infinite series. Later on,

Received 17 April 1968; revised 25 November 1968.