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A NOTE ON ESTIMATING A UNIMODAL DENSITY[1]

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1. Introduction. This paper is concerned with the problem of estimating a unimodal density with unknown mode. Robertson [5] has shown in the case that the mode is known a solution can be represented as a conditional expectation given a σ-lattice. Brunk [2] discusses such conditional expectations as well as other problems.

A σ-lattice, $\mathfrak{L}$, of subsets of a measure space $(\Omega, \mathfrak{A}, \mu)$ is a collection of subsets of Ω closed under countable unions and countable intersections and containing both ϕ and Ω. If Ω is the real line, then the collection of intervals containing a fixed point, m, is a σ-lattice which we shall denote as $\mathfrak{L}(m)$. A function, f, is measurable with respect to a σ-lattice, $\mathfrak{L}$, if the set, $[f > a]$, is in $\mathfrak{L}$ for each real a. In this paper, we shall say a function f is unimodal at M when f is measurable with respect to $\mathfrak{L}(M)$. This definition is equivalent to a more usual definition as seen in the following easily verified remark.

REMARK. A function f is unimodal at M if and only if f is non-decreasing at x for $x < M$ and f is nonincreasing at x for $x > M$.

Let Ω be the real line and let $\mu = \lambda$ be Lebesgue measure. Let L_2 be the set of square integrable functions and $L_2(\mathfrak{L})$ be those members of L_2 which are also measurable with respect to $\mathfrak{L}$. We shall adopt the following definition of conditional expectation with respect to a σ-lattice.

DEFINITION 1.1. If $f \in L_2$, then $g \in L_2(\mathfrak{L})$ is equal to $E(f \mid \mathfrak{L})$, the conditional expectation of f give $\mathfrak{L}$ if and only if

$$\int f\cdot\theta(g)\, d\lambda = \int g\cdot\theta(g)\, d\lambda \tag{1.1}$$

for every θ, a real-valued function such that $\theta(g) \in L_2$ and $\theta(0) = 0$, and

$$\int_A (f - g)\, d\lambda \leqq 0 \tag{1.2}$$

for each $A \in \mathfrak{L}$ with $0 < \lambda(A) < \infty$.

(Brunk [1] shows such a function, g, exists and is unique up to a set of Lebesgue measure 0). In this paper, the estimation of the density will be based upon a strongly consistent estimate of the mode such as those of Nadaraya [3] or of Venter [6]. (The author is aware of a third estimate, as of now unpublished, given by Robertson and Cryer).

2. An estimate of the density. The density, f, to be estimated has mode M which is unknown. Let us assume $y_1 \leqq y_2 \leqq \cdots \leqq y_n$ is the ordered sample of

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