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EXTENSION OF A RESULT OF SENETA FOR THE SUPER-CRITICAL GALTON–WATSON PROCESS

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1. Introduction. Let $Z_0 = 1, Z_1, Z_2 \cdots$ denote a super-critical Galton–Watson process whose non-degenerate offspring distribution has probability generating function $F(s) = \sum_{j=0}^{\infty} s^j \Pr(Z_1 = j)$, $0 \leqq s \leqq 1$, where $1 < m = EZ_1 < \infty$. The Galton–Watson process evolves in such a way that the generating function $F_n(s)$ of Z_n is the nth functional iterate of $F(s)$ and, for the super-critical case in question, the probability of extinction of the process, q, is well known to be the unique real number in $[0, 1)$ satisfying $F(q) = q$. It is the main purpose of this paper to establish the following theorem which gives an ultimate form of the limit result for the case in question.

THEOREM 1. *There exists a sequence of positive constants* $\{c_n, n \geqq 1\}$ *with* $c_n \to \infty$ *and* ${c_n}^{-1} c_{n+1} \to m$ *as* $n \to \infty$ *such that the random variables* ${c_n}^{-1} Z_n$ *converge almost surely to a non-degenerate random variable* W *for which* $\Pr(W = 0) = q$ *and which has a continuous distribution on the set of positive real numbers. Let* s_0 *be any fixed number in* $(0, -\log q)$. *Then,* c_n *can be taken as* $[h_n(s_0)]^{-1}$ *where* $h_n(s)$ *is the inverse function of* $k_n(s) = -\log E\{\exp(-sZ_n)\}$.

This result constitutes an extension of the main result of Seneta [6] where convergence in distribution was established. It should be remarked that, when $EZ_1 = \infty$, it is not possible to find a sequence of positive constants $\{c_n\}$ for which ${c_n}^{-1} Z_n$ converges in distribution to a non-degenerate limit law ([7] Theorem 4.4).

By way of comparison with Theorem 1, we note that:

THEOREM A. (Stigum [8], Kesten and Stigum [3]). *As* $n \to \infty$, $m^{-n} Z_n$ *converges almost surely to a random variable* W_1 *for which* $\Pr(W_1 = 0) = q$ *or* 1 *and which, if* $\Pr(W_1 = 0) < 1$, *has a continuous density on the set of positive real numbers. Moreover, the following two conditions are equivalent:*

(i) $E(Z_1 \log Z_1) < \infty$.
(ii) $\Pr(W_1 = 0) = q$.

Thus, when $E(Z_1 \log Z_1) = \infty$, the norming by m^n is not appropriate and a more subtle norming is required to obtain a non-degenerate limit law. Almost sure convergence in Theorem A is based on the fact (due to Doob) that the process $\{m^{-n} Z_n\}$ is a martingale. The process $\{h_n(s_0) Z_n\}$ is, as was noted in [6], a submartingale but the submartingale convergence theorem is only applicable when $E(Z_1 \log Z_1) < \infty$.

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