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A NOTE ON A CHARACTERIZATION OF THE MULTIVARIATE NORMAL DISTRIBUTION

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1. Introduction. A well-known characterization of the nonsingular p-variate normal distribution is that it is the only jointly dependent multivariate distribution for which the conditional expectation of each variate is a linear function of the remaining $p-1$ variates and the corresponding conditional distribution depends on the remaining $p-1$ variates only through the conditional mean. An illustration of this for the bivariate case is given by Kendall and Stuart, [5] page 352, on the assumption that all moments exist. Féron and Fourgeaud [3] give a concise proof of this characterization. The purpose of this note is to demonstrate that this characterization of the multivariate normal distribution is valid without making any assumptions on moments of higher order than the first. The method of proof reveals a little more detail than was the case with that of Féron and Fourgeaud. This may enable consequences of the main theorem to be more easily determined.

2. The two-vector case. We are interested in two random vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ of n_1 and n_2 dimensions respectively, where $n_1 \geqq 1$ and $n_2 \geqq 1$. We seek to prove the following theorem.

THEOREM 1. *Necessary and sufficient conditions for the joint distribution of* $\mathbf{z}' = (\mathbf{x}_1' : \mathbf{x}_2')$ *to be an* $(n_1+n_2)-$*variate normal distribution are*

(i) $\mathbf{x}_1$ *and* $\mathbf{x}_2$ *are non-degenerate random vectors,*

(ii) *all the first-order absolute moments of* $\mathbf{x}_1$ *and* $\mathbf{x}_2$ *exist,*

(iii) *the conditional distribution of* $\mathbf{x}_j$ *given* $\mathbf{x}_k$ $(j = 1, 2; k = 1, 2$ *and* $j \neq k)$ *depends on* $\mathbf{x}_k$ *only through the conditional mean which is* $E(\mathbf{x}_j \mid \mathbf{x}_k) = \mathbf{a}_j + \mathbf{B}_j'\mathbf{x}_k$ *where each row and column of* $\mathbf{B}_j$ *contains at least one non-zero element,*

(iv) *in* (iii) $\mathbf{B}_1\mathbf{B}_2 \neq \mathbf{I}$ *and* $\mathbf{B}_2\mathbf{B}_1 \neq \mathbf{I}$ *where* $\mathbf{I}$ *is an identity matrix of appropriate order.*

PROOF. We lose nothing in generality if we assume $\mathbf{a}_j = \mathbf{0}$ and $\boldsymbol{\mu}_j = \mathbf{0}$, $j = 1, 2$, where $\boldsymbol{\mu}_j = E(\mathbf{x}_j)$.

The necessity is obvious for the class of distribution that we are considering; see, for example, Rao, [6] page 441. We need only demonstrate the sufficiency of the conditions. For this purpose we note that the characteristic function for $\mathbf{z}$ may be written in two equivalent ways corresponding to the equalities

$$\phi(\mathbf{t}_1, \mathbf{t}_2) = E[\exp(i\mathbf{t}_1'\mathbf{x}_1 + i\mathbf{t}_2'\mathbf{x}_2)]$$

$$= \int_{R_2} \exp(i\mathbf{t}_2'\mathbf{x}_2)\, dF(\mathbf{x}_2) \int_{R_1} \exp(i\mathbf{t}_1'\mathbf{x}_1)\, dF(\mathbf{x}_1 \mid \mathbf{x}_2) \tag{1}$$

$$= \int_{R_1} \exp(i\mathbf{t}_1'\mathbf{x}_1)\, dF(\mathbf{x}_1) \int_{R_2} \exp(i\mathbf{t}_2'\mathbf{x}_2)\, dF(\mathbf{x}_2 \mid \mathbf{x}_1). \tag{2}$$

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