The Annals of Mathematical Statistics
1970, Vol. 41, No. 3, 1074–1077

ON BOUNDS ON THE CENTRAL MOMENTS OF EVEN ORDER OF A SUM OF INDEPENDENT RANDOM VARIABLES

BY BENGT ROSÉN

Uppsala University

1. The theorem. We shall prove the following theorem.

THEOREM. *Let $X_1, X_2, \cdots, X_n$ be independent random variables with mean* 0. *Let p be a natural number and $\lambda_\nu(p)$ and $\rho_\nu(p)$ real numbers such that*

$$(1) \qquad EX_\nu^{2k} \leqq \lambda_\nu^{2k}(p)\rho_\nu(p), \qquad k = 1, 2, \cdots, p, \qquad \nu = 1, 2, \cdots, n.$$

Then

$$(2) \qquad E(\sum_{\nu=1}^n X_\nu)^{2p} \leqq C(p) \max\left((\sum_{\nu=1}^n \lambda_\nu^2(p)\rho_\nu(p))^p, \quad \sum_{\nu=1}^n \lambda_\nu^{2p}(p)\rho_\nu(p)\right)$$

where $C(p)$ is a number which only depends on p.

Before we enter the proof of the theorem, we shall discuss its content somewhat. We list two particular cases, which are included in the theorem.

PARTICULAR CASE 1. *Let $X_1, X_2, \cdots, X_n$ be independent random variables with mean* 0. *Then we have for $p = 1, 2, \cdots$*

$$(3) \qquad E\left|\sum_{\nu=1}^n X_\nu\right|^{2p} \leqq C(p)(\sum_{\nu=1}^n [E|X_\nu|^{2p}]^{1/p})^p.$$

REMARK 1. This is a special case of an inequality due to P. Whittle [4]. Whittle proved that (3) holds for $p \geqq 1$ (also for non-integral p). Whittle also gives a numerical value for $C(p)$.

REMARK 2. By applying Hölder's inequality to the bound in (3), the following inequality is obtained. For $p = 1, 2, \cdots$, we have

$$(4) \qquad E\left|\sum_{\nu=1}^n X_\nu\right|^{2p} \leqq C(p)n^{p-1}\sum_{\nu=1}^n E|X_\nu|^{2p}.$$

This is a special case of a well-known inequality due to Marcinkievitz and Zygmund and Chung, who proved (4) for $p \geqq 1$, see [1] page 348. Whittle's numerical estimate of $C(p)$ works of course in (4) too. Other estimates of $C(p)$ can be found in the paper [2] by Dharmadhikari and Jogdeo.

PARTICULAR CASE 2. *Let $X_1, X_2, \cdots, X_n$ be independent random variables with mean* 0. *Let p be a natural number. Put*

$$\rho_\nu(p) = \max(EX_\nu^2, EX_\nu^{2p}) \qquad \nu = 1, 2, \cdots, n.$$

Then we have for $p = 1, 2, \cdots$

$$(5) \qquad E(\sum_{\nu=1}^n X_\nu)^{2p} \leqq C(p)\max\left((\sum_{\nu=1}^n \rho_\nu(p))^p, \quad \sum_{\nu=1}^n \rho_\nu(p)\right).$$

Received June 9, 1969; revised October 30, 1969.

Institute of Mathematical Statistics is collaborating with JSTOR to digitize, preserve, and extend access to *The Annals of Mathematical Statistics.*
www.jstor.org