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ON THE ASSUMPTIONS USED TO PROVE ASYMPTOTIC NORMALITY OF MAXIMUM LIKELIHOOD ESTIMATES[1]

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1. Introduction. Let Θ be a subset of the real line. For each $\theta \in \Theta$ let p_θ be a probability measure on a certain σ-field $\mathscr{A}$ carried by a set $\mathfrak{X}$. Let $\{\mathfrak{X}^n, \mathscr{A}^n\}$ be the cartesian product of n copies of $\{\mathfrak{X}, \mathscr{A}\}$. Let $P_{\theta, n}$ be the measure product of n copies of p_θ.

Several statistical problems lead to the study of the behavior of the functions $\theta \rightleftarrows P_{\theta, n}$ as n tends to infinity. In particular, the statements concerning maximum likelihood estimates found in Cramér [1] can be considered statements about the local behavior of the logarithm of the likelihood function

$$\Lambda_n(t, \theta) = \log \frac{dP_{t,n}}{dP_{\theta,n}}$$

when n increases indefinitely.

A similar assertion can be made about the deeper results of Wald [11] concerning the asymptotic sufficiency of the maximum likelihood estimates and the asymptotic normality of the family of measures. (Wald gives convergence results uniformly in θ instead of locally; however, the bulk of the argumentation is local.)

The regularity conditions used by Cramér or Wald or other authors, for instance Doob [2], Dugué [3], Wilks [12] always involve the existence of two or three derivatives of the function $t \rightleftarrows dp_t/dp_\theta$ and additional uniform integrability restrictions. These regularity restrictions do not have by themselves any direct or obvious statistical relevance or interpretation. Their role is to permit the proof of the desired theorems.

It has long been realized that the assumptions in question are somewhat too stringent for this purpose and that in fact the asymptotic normality derived from them ought to be a consequence of assumptions involving only first derivatives.

Even if one is not particularly interested in the maximum economy of assumptions one cannot escape practical statistical problems in which apparently "slight" violations of the assumptions occur. For instance the derivatives fail to exist at one point x which may depend on θ, or the distributions may not be mutually absolutely continuous or a variety of other difficulties may occur. The existing literature is rather unclear about what may happen in these circumstances. Note also that since the conditions are imposed upon probability densities they may be satisfied for one choice of such densities but not for certain other choices.

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