The Annals of Mathematical Statistics
1971, Vol. 42, No. 2, 837–838

A CHARACTERIZATION OF THE EXPONENTIAL DISTRIBUTION BY ORDER STATISTICS[1]

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0. Introduction. Several results characterizing the exponential distribution have appeared in the literature in recent years (Basu [1], Crawford [2], Ferguson [3], Govindarajulu [4] and Tanis [6]). Many of these results are based on the independence of suitable functions of order statistics. Here a different type of theorem, which characterizes the exponential distribution, is given and the key idea is to present a function of the order statistics having the same distribution as the one sampled. As a consequence a result on the characterization of a power distribution is obtained.

1. The results. Let X be a random variable with the distribution function $F(\cdot)$. Let $(X_1, X_2, \cdots, X_n)$ be a random sample from F and let $W = \min(X_1, X_2, \cdots, X_n)$.

THEOREM. *If $F(\cdot)$ is a nondegenerate distribution function, then for each positive integer n, nW and X are identically distributed if and only if $F(x) = 1-\exp(-\lambda x)$, for $x \geqq 0$, where λ is a positive constant.*

PROOF. The distribution function of nW is given by

$$F_{nW}(w) = \Pr\{W \leqq w/n\} = 1-[1-F(w/n)]^n. \tag{1}$$

It is easy to verify that $F_{nW}(w) = F_X(w) \equiv F(w)$ when

$$\begin{aligned} F(w) &= 1-\exp(-\lambda w), \qquad & w \geqq 0 \\ &= 0 & w < 0. \end{aligned} \tag{2}$$

The main task is to show that for real w

$$F_{nw}(w) = F(w) \Rightarrow F(w) \qquad \text{is given by (2).} \tag{3}$$

Let $G(w) = 1-F(w)$. Now $F_{nW}(w) = F(w)$ can be expressed as

$$G(nw) = G^n(w) \quad \text{for real} \quad w \quad \text{and integers} \quad n \geqq 1. \tag{4}$$

From (4), it follows that $G(0) = G^n(0)$, and this implies that $G(0) = 0$ or $G(0) = 1$, since $0 \leqq G(w) \leqq 1$. We shall now show that $G(0) = 1$.

To prove this, let us assume that there is a negative number w_0 such that $G(w_0) < 1$. Then, as $n \to \infty$, $G^n(w_0) \to 0$ and $G(nw_0) \to 1$, contradicting (4). Thus, for all $w < 0$, $G(w) \geqq 1$ and hence $G(w) = 1$, for $w < 0$. Since F is nondegenerate we cannot have $G(w) = 0$ for all $w \geqq 0$, which implies that $G(w_1) > 0$

Received June 3, 1970; revised October 12, 1970.

[1]This research was supported in part by National Institutes of Health Grant GM-14031.