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# LIMITING PROPERTIES OF THE MEAN RESIDUAL LIFETIME FUNCTION[1]

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Let $X$ be a positive random variable, with finite mean and infinite essential supremum.

(1) Define, for $0 \leqq x < \infty$, $\quad g(x) = E[X - x \mid X > x]$.

$g$ is called the mean residual lifetime function. We will show that under some general conditions, the conditional distribution of $(X - x)/g(x)$ given that $X > x$ converges as $x \to \infty$ to the standard exponential distribution. These conditions are satisfied, for instance, when $g(x)$ is non-decreasing, while $g(x)/x$ decreases to zero as $x \to \infty$. This will somewhat generalize results in [1]. The case $g(x)/x \to c \in (0, \infty)$ is treated by Feller in [2], page 272. In that case a limiting distribution exists, but is not exponential. The author, ([3]) used these convergence theorems to obtain good bargaining solutions when selling one expensive asset if sampling costs are small. As sampling costs decrease, the seller can permit himself to reject all but the very high offers. The problem is easily solved when the distribution of offers is exponential or the limit obtained by Feller, and its solution in these two cases provides a good asymptotic approximation to the optimal solution (as sampling costs decrease), when the distribution of offers has asymptotically exponential or Feller tails.

For any distribution function $F$, denote $F^*(x) = 1 - F(x)$.

LEMMA 1. *If an absolutely continuous distribution function $F$ on* $[0, \infty)$ *has density $f$, then*

(2) $$F^*(x) = \exp\{-\textstyle\int_0^x (f(t)/F^*(t))\,dt\}, \qquad x \in [0, \infty).$$

PROOF. The two sides agree at $x = 0$, and their logarithms are easily seen to have the same derivatives.

LEMMA 2. *Let $F$ be any distribution function on* $[0, \infty)$, *with finite first moment* $\mu$. *Then*

(3) $$F^*(x) = (\mu/g(x)) \exp\{-\textstyle\int_0^x dt/g(t)\},$$

($g$ was defined in (1)).

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