Comment

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As someone who has been working in the area of chaos and statistics for the past few years, I find it very gratifying to witness the beginnings of growing interest among statisticians in this field. As noted by the authors of both papers, the area of chaos has grown very rapidly, spilling over into many different disciplines (notably the physical and biological sciences) and generating much controversy and a wealth of ad hoc techniques (many statistical in nature) for the analysis of chaotic data. The authors of both papers are to be commended for providing stimulating overviews of the theory, techniques and applications of chaos, a far from easy task given the explosive growth of literature in this area.

There is considerable opportunity for statisticians to make an impact in this field (by supplying practitioners, usually scientists, with appropriate methodology), as well as to make an impact on statistics itself (by incorporating features of chaos, such as nonlinear deterministic models, into data analysis). Recent statistical work in chaos (on topics ranging from estimation of dimension and Lyapunov exponents to nonlinear prediction) include Denker and Keller (1986), Cutler and Dawson (1989, 1990), Nychka, McCaffrey, Ellner and Gallant (1990), Wolff (1990), Berliner (1991), Bøgsted Hansen (1991), Cheng and Tong (1991), Cutler (1991) and Smith (1991, 1991b). I hope this issue of *Statistical Science* will encourage more statisticians to consider future work in this area.

Numerous topics are addressed by both papers. I will limit my discussion to a few areas in which I have experience or special interest.

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At least one major source of difficulty in statistical analyses of chaotic systems is the demand by practitioners for techniques that are applicable to an enormous variety of dynamical systems. Whereas "chaos" itself has certain specific defining properties (such as sensitive dependence on initial conditions), the types of probability distributions arising from chaotic models do not. Here, I assume an ergodic system with probability measure P arising as the limiting empirical distribution along a trajectory (corresponding to a particular set of initial conditions). Specifically

$$(1) \qquad P(B) = \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} I_B(T^k(x)) \qquad \text{for } P\text{-almost all } x$$

where T is the map being iterated. The attractor of a chaotic system need not be fractal (e.g., the attractor of the logistic map $T(x) = 4x(1-x)$ is the unit interval [0, 1]), although a fractal structure is a frequent feature. (In the next section, I will indicate how the presence or absence of an underlying fractal structure can affect dimension estimation procedures.) The distribution P, which is supported on the attractor, may have very different behaviors for different systems (indeed, even for different parameter values of the same system). In the case where the attractor A is a smooth subset (e.g., a manifold) of $\mathbf{R}^N$, P may be absolutely continuous with respect to the Lebesgue measure on A, thus possessing an invariant density g. This density g may or may not be bounded. For example, in the logistic case with $a = 4$, the density $g(x) = \pi^{-1}[x(1-x)]^{-1/2}$ has singularities at both 0 and 1, whereas the "tent" map (discussed by Chatterjee and Yilmaz) features the uniform distribution as invariant measure. The presence of singularities in the density (or even regions of bounded yet steep density) can adversely affect estimation procedures. More typically, at least for dissipative systems in higher dimensions, the attractor is a fractal and, thus, P is necessarily singular with respect to Lebesgue measure. It is interesting that, in the area of chaos, continuous singular distributions arise as natural objects of study, rather than as examples of mathematical pathology (as frequently portrayed in mathematical statistics courses). In fact, due to the possible complexity of attracting sets and distributions, a strong interplay between mathematics and statistics is often required to analyze a chaotic system properly. In practice, the governing equations of a system are rarely known. [Even when they are, it is often difficult or impossible to prove rigorous results about the system asymptotics. The "simple" two-dimensional Hénon mapping defied rigorous analysis from 1976 until

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