Lundy and Kruskal, 1985; Kruskal, Harshman and Lundy, 1989; Lundy, Harshman and Kruskal, 1989).

Krijnen and Ten Berge (1991) developed variants of the basic PARAFAC algorithm to put nonnegativity constraints on the solution by using special least squares regression algorithms from Lawson and Hanson (1974). Durrell et al. (1990) refer to programs for three-way and four-way PARAFAC models (Lee, 1988) which also included nonnegativity constraints.

3.6 Additional Issues

In the above sections, the general focus has been on models and algorithms, but there are several issues in connection with these models which have not been mentioned so far. Very prominent, for instance, in Harshman's work, has been the question of preprocessing (i.e., centering and standardisation) of the data before the three-way analysis. Harshman and Lundy (1984b) discuss this issue in great detail touching on both algebraic and practical aspects (see also Kroonenberg, 1983). Ten Berge and Kiers (1989) and Ten Berge (1989) provide some theoretical results with respect to the iterative centering and standardisation proposed by Harshman and Lundy.

Another issue in this context is the postprocessing of output, that is, representation, graphing and transformations of the basic output of the programs to enhance interpretability (see especially Harshman and Lundy, 1984b; Kroonenberg, 1983).

Smilde (1992) raises the issue of variable selection for three-way data, as well as the problem of nonlinearities in the data and their effect on the solutions. These issues can also be seen as a serious concern in such areas like ecology where nonlinearities are the rule rather than the exception (see, e.g., Faith, Minchin and Belbin, 1987).

A final point is that within the framework of the analysis of covariance structures, McDonald (1984) has discussed the PARAFAC model, cited its limitations and proposed an altogether different (stochastic) approach to the kind of three-way data psychologists often encounter.

4. CONCLUSION

With the above comments, I have attempted to give a rough outline of research on the PARAFAC model. The model itself is only one of several conceivable models for three-way data, but a fully fledged exposé is not feasible here. What makes the PARAFAC model special is that it has a unique solution, a situation which is fairly unique in three-way land.

Comment

Donald S. Burdick

Multilinear models are fascinating because of the richness of their mathematical structure and the usefulness of their applications. The authors have done a fine job of presenting both of these features. I welcome their paper and hope that it has the effect of stimulating interest in this important topic.

Having said that, I must add my opinion that it is a mistake to shy away from tensors. The geometry of tensor products can be a source of valuable insight when struggling with the complicated details of multilinear algebra. The geometric perspective is especially useful when trying to make sense out of the nonuniqueness that occurs when model parameters are not identifiable.

For example, the concept of tensor products of vector spaces can shed light on the structure of the T3 model. Let $\boldsymbol{Y}$ denote an $I \times J \times K$ data array and write

$$\boldsymbol{Y} = \mu + \mathrm{e}$$

where μ is given by (19). The data array $\boldsymbol{Y}$ is unconstrained, which is tantamount to saying that $\boldsymbol{Y}$ is an arbitrary vector in $R^I \otimes R^J \otimes \mathrm{R}^K$, the tensor product of real Euclidean spaces of dimensions I, J and K, respectively. The array μ, however, is constrained by expression (19). What is the nature of that constraint? Expression (19) stipulates that μ lie in $\mathcal{A} \otimes \mathcal{B} \otimes \mathcal{C}$, where $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are the respective subspaces of R^I, R^J and R^K spanned by the columns of $\boldsymbol{A}$, $\boldsymbol{B}$ and $\boldsymbol{\Gamma}$, respectively. The least squares fit of μ to $\boldsymbol{Y}$ is the projection of $\boldsymbol{Y}$ on $\mathcal{A} \otimes \mathcal{B} \otimes \mathcal{C}$. From a geometric perspective, the nonidentifiability is obvious, because the projection of a data vector on a subspace is unaffected by changes in the basis spanning the subspace. Replacing $\boldsymbol{A}$ by $\boldsymbol{AM}$ amounts to no more than a change of basis for $\mathcal{A}$.

Donald S. Burdick is Associate Professor, Institute of Statistics and Decision Sciences, Duke University, Durham, North Carolina 27706.