METHODS OF OBTAINING PROBABILITY DISTRIBUTIONS[1]

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The emphasis of this paper will be on method. Special results will be cited in order to illustrate the methods rather than to summarize achievement in the field; for that has been done already by Rider (1930, 1935) Irwin (1935) and Shewhart (1933) in recent surveys. The purpose is to describe and to illustrate most of the methods that have been used to determine exact probability distributions, and to show that they are all derivable from one fundamental theorem. In order to prove this unity in a simple manner, it will be desirable to omit from consideration methods which are essentially ingenious forms of counting, such as are used in sampling without replacements from finite universes, and in finding the sampling distribution of a percentile.

The general problem to be discussed may be stated as follows: N individuals $(t_1, \cdots, t_N)$ are drawn, one at a time with replacements, from a universe whose probability distribution is $\phi(t)$. A certain single valued function of the t's is formed. This is called a parameter of the sample, and is frequently also, but not necessarily, a useful estimate of the corresponding parameter of the universe. The problem is to find its probability distribution, $f(x)$. As usual, a probability distribution is a function which is required to be defined, except perhaps at a set of measure zero, throughout the infinite domain of its variables; it is nowhere negative, and its integral over its domain is unity.

Most of the more recent developments of the theory relate to a more general form of this problem. Instead of N individuals, there are N sets of n individuals in each set, and these sets are drawn respectively from $M(M \leqq N)$ universes, each of which is described by a function of n independent variables, thus:

$$\phi^{(i)}(t_1, \cdots, t_n); (i = 1, \cdots, M). \tag{1}$$

Instead of a single parameter there are P parameters, and each is a single valued function of the observed values of the nN individuals in the sample, thus:

$$x_i = g_i(t_1^{(1)}, \cdots, t_n^{(1)}; \cdots; t_1^{(N)}, \cdots, t_n^{(N)}); (i = 1, \cdots, P) \tag{2}$$

The first method to be described is fundamental and will be designated as

Theorem I. Let it be required that each g as described in (2) be not only single valued but also constant at most in a set of measure zero in the nN-way space of the t's. Then

$$\int_p f(x_1, \cdots, x_P)\, dX = \int_q \phi(t_1^{(1)}, \cdots, t_n^{(N)})\, dT \tag{I}$$

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