NOTES

This section is devoted to brief research and expository articles, notes on methodology and other short items.

NOTE ON THE LAW OF LARGE NUMBERS AND "FAIR" GAMES

By W. Feller

Cornell University

1. "Fair" games. Let \( \{X_k\} \) be a sequence of independent random variables with the same cumulative distribution function \( V(x) \). Suppose that the expectation

\[
E(X_k) = \int_{-\infty}^{+\infty} x \, dV(x) = M
\]

exists, and put

\[
S_n = X_1 + \cdots + X_n.
\]

The weak law of large numbers states\(^1\) that for every \( \epsilon > 0 \) and \( n \to \infty \)

\[
\Pr \{ |S_n - nM| < \epsilon n \} \to 1.
\]

In the picturesque language of the theory of games this means that, after a large number of trials, the accumulated gain \( S_n \) will, with great probability, be of the order of magnitude of \( nM \). This led to the definition that a game is "fair" if the entrance fee for each trial is \( M \). Unfortunately this definition creates the erroneous notion that a "fair" game is necessarily fair. To disprove it we shall (section 3) exhibit an example which will show:

(1) A game can be "fair" and nevertheless such that the probability tends to one that, after \( n \) trials, the player will have sustained a loss \( L_n = nM - S_n \) of the order of magnitude \( n(\log n)^{-\eta} \), where \( \eta > 0 \) is arbitrarily small. In other words, in our example

\[
\Pr \{|nM - S_n > (1 - \epsilon)n(\log n)^{-\eta}\} \to 1.
\]

Of course, \( L_n \) is necessarily of smaller order of magnitude than \( n \); however, our example can be modified in such a way that the ratio of the loss \( L_n \) to the accumulated entrance fees \( nM \) decreases as slowly as one pleases.

This shows that a "fair" game can be exceedingly disadvantageous. Conversely, an "unfair" game can very well be advantageous. If a careful driver insures his car, the game is clearly "unfair" according to definition, and yet some

\(^1\) Usually (3) is proved only under more restrictive hypotheses. Actually the finiteness of \( E(X_k) \) implies even the strong law of large numbers; cf. Kolmogoroff, Grundbegriffe der Wahrscheinlichkeitsrechnung (Berlin 1933), p. 59.