The Annals of Mathematical Statistics
1968, Vol. 39, No. 5, 1762–1763

AN INEQUALITY FOR THE RATIO OF TWO QUADRATIC FORMS IN NORMAL VARIATES

By Koteswara Rao Kadiyala

The University of Western Ontario

The distribution of ratios of quadratic forms has been investigated by many authors. Two simple inequalities for the ratio of quadratic forms in independent normal variates are presented.

Theorem. *If* $\lambda_1 \geqq \lambda_2 \geqq \cdots \geqq \lambda_n > 0$, $d_1 \geqq d_2 \geqq \cdots \geqq d_n > 0$, *and* $Z_1, \cdots, Z_n$ *are positive random variables, then*

$$\text{(i)} \qquad P\{\textstyle\sum_1^n \lambda_i d_i Z_i / \sum_1^n d_i Z_i \leqq \nu\} \leqq P\{\sum_1^n d_i Z_i / \sum_1^n Z_i \leqq \nu\}.$$

If $X_1, \cdots, X_n$ *are independent* $N(0, 1)$ *random variates, and* $(d_1^*, \cdots, d_n^*)$ *is any rearrangement of* $d_1, \cdots, d_n$, *then*

$$\text{(ii)} \quad P\{\textstyle\sum_1^n \lambda_i d_i^* X_i^2 / \sum_1^n d_i^* X_i^2 \leqq \nu\} \leqq P\{\sum_1^n \lambda_i d_{n-i+1} X_i^2 / \sum d_{n-i+1} X_i^2 \leqq \nu\}.$$

Proof. (i) The proof[1] follows from the fact that

$$\textstyle\sum_1^n \lambda_i d_i Z_i / \sum_1^n d_i Z_i \geqq \sum_1^n \lambda_i Z_i / \sum_1^n Z_i,$$

which follows as a special case of an inequality due to Tchebychef [1], p. 168. It is clear, of course, that, if the d's are increasingly ordered, the inequality in (i) would go the opposite way.

$$\text{(ii)} \qquad P\{\textstyle\sum_1^n \lambda_i d_i^* X_i^2 / \sum_1^n d_i^* X_i^2 \leqq \nu\}$$
$$= P\{\textstyle\sum_1^n (\lambda_i - \nu) d_i^* X_i^2 \leqq 0\} = (2\pi)^{-n/2} \int_E e^{-1/2x'x}\, dx,$$

where E is the set appearing on the left hand side of (ii). Now if $d_1^* \neq d_n$ there exists a k such that $d_k^* < d_1^*$. And,

$$P\{\textstyle\sum_1^n (\lambda_i - \nu) d_i^* X_i^2 \leqq 0\}$$
$$= P\{(\lambda_1 - \nu_1) d_1^* X_1^2 + (\lambda_k - \nu) d_k^* X_k^2 \leqq \chi_0\} / P(\chi \leqq \chi_0)$$

where $\chi = -\sum_{i \neq 1,k} (\lambda_i - \nu) d_i^* X_i^2$. But,[2]

$$\text{(1)} \quad P\{(\lambda_1 - \nu) d_1^* X_1^2 + (\lambda_k - \nu) d_k^* X_k^2 \leqq \chi_0\}$$
$$= (2\pi)^{-1} \int\int_{E_0} e^{-\frac{1}{2}(x_1^2 + x_2^2)}\, dx_1\, dx_2 \leqq (2\pi)^{-1} \int\int_{E_1} e^{-\frac{1}{2}(x_1^2 + x_2^2)}\, dx_1\, dx_2$$

where $E_0 = \{x_1, x_2 : (\lambda_1 - \nu) d_1^* x_1^2 + (\lambda_k - \nu) d_k^* x_2^2 \leqq \chi_0\}$

and $E_1 = \{x_1, x_2 : (\lambda_1 - \nu) d_k^* x_1^2 + (\lambda_k - \nu) d_1^* x_2^2 \leqq \chi_0\}$.

Received 15 August 1967.

[1] The author is indebted to the referee who suggested this proof which is more direct and simpler than the original proof.

[2] If $(\lambda_1 - \nu) \geqq 0$ and $(\lambda_k - \nu) \leqq 0$ the fact that the inequality in (1) follows is trivial. If both $(\lambda_1 - \nu)$ and $(\lambda_k - \nu)$ are strictly positive (1) follows from the fact that the areas of the two sets E_0 and E_1 are equal.