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THE MAXIMUM VARIANCE OF RESTRICTED UNIMODAL DISTRIBUTIONS

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1. Introduction. The least upper bound is derived for the variance of unimodal probability distributions, which are restricted to a finite interval of the real line and possess probability densities with respect to Lebesgue measure. In applications the probability of some rare event is often desired, where the exact form of a distribution and/or its variance are not readily derivable, although the distribution is intuitively known to be unimodal. In such cases an upper bound for the desired probability may be available, e.g., via Chebycheff's inequality, as a function of an upper bound upon the unknown variance. Variance bounds also find use in applications of the Central Limit Theorem. Outside of such applications, it is of separate academic interest to observe the extent to which the condition of unimodality limits the attainable variance.

Johnson and Rogers [2] have shown that for unimodal distributions, the variance is bounded below by $(\text{mean-mode})^2/3$. More recently, Gray and Odell [1] have shown that the variance of certain piecewise continuous functions, restricted to a finite interval, is maximized if taken with respect to the uniform density compared with any other density on the interval that is unimodal, piecewise continuous, and symmetric about the interval midpoint. Their result indicates that the uniform density has the maximum variance within the cited class of symmetric densities.

This paper extends the results of Gray and Odell by dropping the requirement of symmetry–however, at the expense of restricting attention to the distribution variance. Obviously, the amount of available description of the distribution determines the exactitude of variance bounds. Merely knowing that a distribution is restricted to $[a, b]$ serves to bound its variance by $(b - a)^2/4$, which derives from the non–unimodal Bernoulli distribution with atoms of equal probability measure at $x = a$ and $x = b$. It will be shown here that the requirement of unimodality restricts the variance to $(b - a)^2/9$, and that this is a least upper bound. Note that this bound exceeds the variance $(b - a)^2/12$ of a uniform density on $[a, b]$. Some sufficient conditions are also given for the distribution variance not to exceed $(b - a)^2/12$. The moments of the distribution, as with the distribution itself, are presumed unknown.

2. Preliminaries. Let C^* denote the class of probability densities with respect to Lebesgue measure on the real line, that are restricted to a finite interval $[a, b]$, and that are unimodal. Let $C = \{f \varepsilon C^*; \text{some modes of } f \text{ are in the interior } (a, b)\}$. If $x = m$ is a mode of $f(x)$, then $f(x)$ is monotone on $[a, m)$ and on $(m, b]$ and can

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