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On Toeplitz forms and stationary processes

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1. Introduction

This paper is intended to show that the theory of Toeplitz forms is applicable to some important linear statistical problems concerning stationary stochastic processes with a discrete time-parameter. Although the main results on Toeplitz forms have been known for more than 30 years they do not seem to have attracted the attention they deserve of the mathematical statisticians. This motivates a short summary of the most important results in this theory.

That summary is given in the next section. In **3** we study the problem of prediction in the light of what has been said in **2** about Toeplitz forms. Section **4** is devoted to the problem of finding asymptotic expressions for the distribution of certain quadratic forms which is of importance in the statistical analysis of stationary time series. In **5**, finally, we study the estimation of the mean value of the process and a similar but more general problem of estimation which i.a. contains a theorem of KOLMOGOROFF on interpolation of stationary processes as a special case.

2. Some results from the theory of Toeplitz forms

2.1. Consider a Hermitian matrix M of the form

$$M = \begin{Bmatrix} c_0 & c_{-1} & c_{-2} & \dots & c_{-n} \\ c_1 & c_0 & c_{-1} & \dots & c_{-n+1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ c_n & c_{n-1} & c_{n-2} & \dots & c_0 \end{Bmatrix} = \{ c_{\nu-\mu} \}; \quad \nu, \mu = 0, 1, \dots n. \tag{1}$$

Then M is said to be a *Toeplitz matrix* and the adjoint quadratic form

$$T = \sum_{\nu, \mu=0}^{n} c_{\nu-\mu} x_\nu \bar{x}_\mu$$

is called a *Toeplitz form*. An important class of Toeplitz forms is defined in the following way. Let $f(\lambda)$ be a non-negative Lebesgue-integrable function defined in the interval $(-\pi, \pi)$, and let the matrix elements be defined as