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On the linear prediction problem for certain stochastic processes

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1. Consider an infinite sequence of complex-valued random variables

$$\ldots, x_{-2}, x_{-1}, x_0, x_1, x_2, \ldots$$

with finite second order mean values. For the sake of simplicity, we shall assume throughout that all first order mean values of the x_n reduce to zero, while there is at least one x_n having a variance different from zero:

$$E\,x_n = 0 \text{ for all } n,$$

$$E\,|x_n|^2 > 0 \text{ for some } n.$$

The *covariance function* of the x_n sequence is

$$R(m, n) = E\,(x_m \overline{x_n}).$$

We may interpret x_n as a measure of the state of some observed variable system at the time point nt, where t is a given quantity. The sequence of the x_n, with $n = \cdots, -1, 0, 1, \ldots$, will then represent the temporal development of this system, and will constitute a *stochastic process with discrete time.* In the sequel, we shall always take $t = 1$, so that the subscript n may be directly regarded as measuring time.

The *prediction problem* for a process of this kind is the problem of predicting the state of the process at some future time point, when its past development is assumed to be more or less known. In this paper we shall only be concerned with *linear least squares prediction.* Thus we shall want to find the "best possible" prediction of a certain x_n by means of linear operations acting on certain variables belonging to the past of the process, interpreting the "best possible" in the sense of minimizing the mean value of the squared error of prediction.

Consider the expression

$$\min_{c_0, \ldots, c_q} E\,|x_n - c_0 x_{n-p} - c_1 x_{n-p-1} - \cdots - c_q x_{n-p-q}|^2 = s^2_{npq} \geqq 0,$$

where n, p and q are fixed integers with $p > 0$, $q > 0$, while the minimum has to be taken for all complex quantities $c_0, \ldots, c_q$. Then s_{npq} will be the least possible error of prediction, when x_n has to be linearly predicted in terms of x_{n-p}, x_{n-p-1},