1.63091 Communicated 22 May 1963 by Åke Pleijel and Lars Gårding

A generalization of a Tauberian theorem by Pleijel

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1. Introduction

In [4], [5] Pleijel proved the following Tauberian theorem. (The factor $|\lambda|^{-h}$ in [4] is here included in the measure $d\sigma$.)

If $s<1$, $\sigma(\lambda)\in T^s$, $\sigma(-\lambda)\in T^s$ *and*

$$\int_{-\infty}^{+\infty}(\lambda+t)^{-1}\,d\sigma(\lambda)=t^{-1}p(t^{-1})+o(t^{s-1}), \tag{1}$$

when $t\to\infty$ *along all non-real halfrays from the origin, then* $\sigma(\lambda)\in I^s$ *and* $\sigma(-\lambda)\in I^s$. *If* s *is an integer,* $I^{-s}(\sigma(\lambda)-(-1)^s\sigma(-\lambda))\in\omega^0$.

In (1) p is a real polynomial, and the relation $\varphi\in\omega^s$ means that $\varphi(\lambda)=$ constant $+\,o(\lambda^s)$ when $\lambda\to+\infty$. I^k is defined by $dI^k\varphi=\lambda^k d\varphi$. If $I^k\varphi\in\omega^{k+s}$ for one value of k, then the same relation is valid for all $k\neq -s$ and we write $\varphi\in I^s$. When $s\neq 0$, $I^s=\omega^s$, but ω^0 is a proper subset of I^0. The integral in (1) is convergent if and only if $I^{-1}\sigma\in\omega^0$. Finally, $\varphi\in T^s_+(T^s_-)$ means that there is a constant C such that $d\varphi+C\lambda^{s-1}d\lambda$ is $\geqslant 0\ (\leqslant 0)$ for sufficiently large positive λ, and $T^s=T^s_+\cup T^s_-$.

In the proof in [4] of the Tauberian theorem, the asymptotic relation (1) is used only along the imaginary halfrays except in two exceptional cases, which occur when s is an integer. In these cases (1) is used for the four halfrays $\arg t=\frac{1}{4}\pi+n\cdot\frac{1}{2}\pi$, $n=0,1,2,3$. Thus the conditions in the theorem are unnecessarily strong.

A closer study of the question has shown that it is sufficient to have the asymptotic relation along an arbitrary pair of non-real halfrays which are separated by the imaginary axis. This includes the case when one or both halfrays are purely imaginary. Moreover, the use of a larger Tauberian class than T^s led to a more natural proof of the Tauberian theorem.

We can suppose that the halfrays are the imaginary halfrays. For if (1) is valid along another pair of rays, the relation holds also along the complex conjugate rays as the polynomial p is real. The integral and the polynomial in (1) increase so slowly when $t\to\infty$ that the Phragmén–Lindelöf principle can be used in the angles formed in the upper and in the lower halfplane by the four halfrays. These angles contain the imaginary halfrays.