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Generalized zeta-functions

By Burton Randol

1. Introduction

In this paper, we will show that some of the features common to certain zeta-functions which occur in analysis are properties of much more general objects. As it turns out, a considerable unity of treatment can be achieved with surprisingly mild assumptions, and when possible, we will point out how familiar cases fall within the more general framework. Our starting point will be Bochner's paper [1], and I would like here to express my gratitude to Professor Bochner for his generous advice during the preparation of this paper, and to thank Professors Edward Nelson and Robert Langlands for several informative conversations.

2. Some examples

1. The Riemann zeta-function can be defined in a right half-plane by a Dirichlet series

$$2\zeta(2s) = \Sigma' n^{-2s},$$

where the prime indicates that the summation is over all non-zero integral lattice-points in E^1. In particular, the series is of the general type

$$\Sigma'[P(n)]^{-s},$$

where P is a positive form in E^1.

2. In [3], P. Epstein discussed, among other things, series of the type

$$\Sigma'[P(n)]^{-s},$$

where P is a positive-definite quadratic form in E^k, and where the summation is over all non-zero integral lattice-points in E^k. He was able to show that the function corresponding to the series (the latter clearly converges uniformly in some right half-plane) is meromorphic in s, and satisfies a functional equation analogous to that for the Riemann zeta-function.

3. Bochner [1] considered series of the type

$$\Sigma'[P(n)]^{-s},$$