1.65 11 2 Communicated 8 September 1965 by H. Cramér and O. Frostman

Semi-groups of isometries and the representation and multiplicity of weakly stationary stochastic processes

By G. Kallianpur and V. Mandrekar

1. Introduction and Summary

It is the purpose of this paper to examine the work of J. L. B. Cooper on the representation of a continuous semigroup $\{S_t, t\geqslant 0\}$ of isometries on a separable Hilbert space $\mathcal{Y}$ and to show how it can be adapted to give a complete discussion of the representation theory of a very general class of continuous parameter, weakly stationary stochastic processes which include finite as well as infinite dimensional processes [12]. The possibility of such a connection between Cooper's results and representations of stationary processes has been noted by P. Masani and J. Robertson [15] (also [14]). However, the approach of these authors has been to reduce the study of continuous parameter processes to certain discrete parameter processes associated with them ([15], § 4).

The point of view adopted in this paper enables us to dispense with the associated discrete parameter process and to give a time domain analysis based directly on the stochastic process itself. A significant tool in our analysis is the fundamental notion of multiplicity of a stochastic process introduced recently by H. Cramér [2] and T. Hida [10], and studied extensively by the former author in subsequent papers ([3], [4]). Before we can bring out the relevance of Cooper's ideas to our present aims, it is necessary to complete his basic result in two essential respects: firstly, to introduce the definition of multiplicity of Cooper's representation and secondly to show that it is equal to the dimension of the deficiency subspace $R^{\perp}$, R being the range of the Cayley transform V of the maximal symmetric operator H, where iH is the infinitesimal generator of $\{S_t\}$. Cooper's result thus completed and amplified is presented as Theorem 2.1 in section 2.

In sections 3 and 4 we obtain some interesting points of contact with more recent work on isometric operators in Hilbert space. We show in section 3 that Theorem 2.1 immediately yields in a simple and natural way a direct integral representation in terms of "differential innovation" subspaces obtained earlier by Masani [14]. Indeed, the vector valued integral of [14] turns out to be nothing other than the orthogonal sum of N "stochastic integrals", N being the multiplicity of the representation. Section 4 carries the study of the differential innovation subspaces further. Each such subspace is shown to be a "weighted" orthogonal sum of $V^n(R^{\perp})\,(n=0,1,\ldots)$ which are the innovation subspaces of the associated discrete representation (1.1) of [14]. We believe that this theorem (Theorem 4.1) puts in better perspective, the intrinsic relationship between the given continuous parameter process and its associated discrete parameter process.