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On the complexity of algebraic numbers, II. Continued fractions

by

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1. Introduction

Let $b\geqslant 2$ be an integer. Émile Borel [9] conjectured that every real irrational algebraic number α should satisfy some of the laws shared by almost all real numbers with respect to their b-adic expansions. Despite some recent progress [1], [3], [7], we are still very far away from establishing such a strong result. In the present work, we are concerned with a similar question, where the b-adic expansion of α is replaced by its sequence of partial quotients. Recall that the continued fraction expansion of an irrational number α is eventually periodic if and only if α is a quadratic irrationality. However, very little is known regarding the size of the partial quotients of algebraic real numbers of degree at least three. Because of some numerical evidence and a belief that these numbers behave like most of the numbers in this respect, it is often conjectured that their partial quotients form an unbounded sequence, but we seem to be very far away from a proof (or a disproof). Apparently, Khinchin [16] was the first to consider such a question (see [4], [27] and [29] for surveys including a discussion on this problem). Although almost nothing has been proved yet in this direction, some more general speculations are due to Lang [17], including the fact that algebraic numbers of degree at least three should behave like most of the numbers with respect to the Gauss–Khinchin–Kuzmin–Lévy laws.

More modestly, we may expect that if the sequence of partial quotients of an irrational number α is, in some sense, 'simple', then α is either quadratic or transcendental. The term 'simple' can of course lead to many interpretations. It may denote real numbers whose continued fraction expansion has some regularity, or can be produced by a simple algorithm (by a simple Turing machine, for example), or arises from a simple dynamical system ... . The main results of the present work are two new combinatorial transcen-

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