

CONCERNING A THEOREM OF PALEY ON LACUNARY POWER SERIES

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1. The main purpose of this paper is to prove the following theorem:

THEOREM 1. *If a lacunary power series*

$$\sum_{k=1}^{\infty} c_k z^{n_k} \quad (n_{k+1}/n_k > q > 1) \quad (1.1)$$

$$\text{satisfies the conditions} \quad c_k \rightarrow 0, \quad \sum |c_k| = \infty \quad (1.2)$$

and if ζ is any point in the complex plane, then there exists a point ξ on the unit circle such that $\sum c_k \xi^{n_k}$ converges to ζ .

This theorem was stated by Paley in a note [1]. Since the note was in the nature of a research announcement no proof was given for the theorem. However, in a letter to Prof. Zygmund dated Oct. 7, 1932, Paley gave an outline of the proof, as he saw it, for his theorem. The argument is made to depend on a lemma (Theorem 2 of the present paper). Paley believed that the proof of the lemma would follow the reasoning given in Zygmund's article [4]. However, attempts to reconstruct the proof along these lines have not succeeded. Taking this lemma for granted, Paley next sketches how his theorem may be deduced from it. He presents an ingenious idea how this part of the argument is to be carried out. Paley's idea here is, as it turns out, completely successful, but the details which need to be supplied are lengthy.

The purpose of the present paper is, therefore, twofold. First, a proof is given for theorem 2. The argument, which is contained in sections 1–4, is rather complex, and seems to indicate that the simpler idea envisaged by Paley could not succeed. It should be noted that when $q > 3$, Theorem 2 can be given a very simple proof.