A PRIORI INEQUALITIES CONNECTED WITH SYSTEMS OF PARTIAL DIFFERENTIAL EQUATIONS

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Introduction

In the more recent development of the theory of general (linear) partial differential equations, the so-called a priori inequalities play a prominent role. For instance, the important comparison of two partial differential operators $P(D)$ and $Q(D)$ with constant coefficients, studied by L. Hörmander [1], depends on the existence of a constant C such that

$$\| Q(D)u \| \leqslant C \| P(D)u \|$$

for all functions $u = u(x)$ of class $\mathcal{D}(\Omega)$. (The norms are L^2-norms with respect to Lebesgue measure in a given region Ω in a Euclidean space R^n. The class $\mathcal{D}(\Omega)$ consists of all infinitely differentiable functions of compact support in Ω.) One of Hörmander's basic results [1, Theorem 2.2] asserts that, if Ω is bounded, such a constant C exists if and only if the ratio $\tilde{Q}(\xi)/\tilde{P}(\xi)$ remains bounded as a function of $\xi \in R^n$. Here $\tilde{P}(\xi)$ denotes a certain "norm function" associated with the polynomial $P(\xi)$ in terms of which the operator $P(D)$ is defined (cf. § 1 below).

The present paper is concerned with similar problems for *systems* of differential operators. Such a system is conveniently described as a matrix $\mathbf{P}(D)$ whose elements are partial differential operators $P_{ij}(D)$. If $\mathbf{Q}(D)$ denotes another such matrix, we shall find a necessary and sufficient condition for the existence of a constant C such that

$$\| \mathbf{Q}(D)\mathbf{u} \| \leqslant C \| \mathbf{P}(D)\mathbf{u} \|$$

for all column vectors $\mathbf{u} = \mathbf{u}(x)$ whose elements $u_j(x)$ are of class $\mathcal{D}(\Omega)$. (Theorems 3.1 and 4.) In § 5 we treat a more general problem of the same nature, viz., to