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Picard potentials and Hill's equation on a torus

by

FRITZ GESZTESY and RUDI WEIKARD

University of Missouri
Columbia, MO, U.S.A.

University of Alabama
Birmingham, AL, U.S.A.

1. Introduction

Hill's equation has drawn an enormous amount of consideration due to its ubiquity in applications as well as its structural richness. Of particular importance in the last 20 years is its connection with the KdV hierarchy and hence with integrable systems.

We show in this paper that regarding the independent variable as a complex variable yields a breakthrough for the problem of an efficient characterization of all elliptic finite-gap potentials, a major open problem in the field. Specifically, we show that elliptic finite-gap potentials of Hill's equation are precisely those for which all solutions for all spectral parameters are meromorphic functions in the independent variable, complementing a classical theorem of Picard. The intimate connection between Picard's theorem and elliptic finite-gap solutions of completely integrable systems is established in this paper for the first time.

In addition, we construct the hyperelliptic Riemann surface associated with a finite-gap potential (not necessarily elliptic), i.e., determine its branch and singular points from a comparison of the geometric and algebraic multiplicities of eigenvalues of certain auxiliary operators associated with Hill's equation. These multiplicities are intimately correlated with the pole structure of the diagonal Green's function of the operator $H=d^2/dx^2+q(x)$ in $L^2(\mathbf{R})$. Our construction is new in the present general complex-valued periodic finite-gap case.

Before describing our approach in some detail, we shall give a brief account of the history of the problem involved. This theme dates back to a 1940 paper of Ince [43] who studied what is presently called the Lamé–Ince potential

$$q(x)=-g(g+1)\mathcal{P}(x+\omega_3), \quad g\in\mathbf{N},\ x\in\mathbf{R}, \tag{1.1}$$

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