COMMUTATORS, PERTURBATIONS, AND UNITARY SPECTRA

BY

C. R. PUTNAM

Purdue University, Lafayette, U.S.A. (1)

1. Introduction. Let A and B denote linear operators, bounded or unbounded, on a Hilbert space H of elements x. As is customary, let $\|x\| = (x, x)^{\frac{1}{2}}$ and put $\|A\| = \sup\|Ax\|$, where $\|x\| = 1$. If A and B are bounded and if C denotes the commutator of A and B,

$$C = AB - BA, \tag{1.1}$$

then it is well known that

$$\|C\| \leqslant 2\|A\|\,\|B\|, \tag{1.2}$$

and that the inequality cannot be improved by replacing the 2 by $2 - \varepsilon$ with $\varepsilon > 0$. Indeed, simple examples with finite matrices $A \neq 0$, $B \neq 0$ and A, iB (hence also C) even self-adjoint show that the equality of (1.2) may hold.

Part I of this paper will be concerned with an improvement of (1.2) when B is bounded but otherwise arbitrary, A and C are bounded and self-adjoint, *and C is non-negative.* If the space H is finite-dimensional this last restriction forces C to be 0, since the trace of C, which equals the sum of its eigenvalues, is 0. On the other hand, in the infinite dimensional case, examples show that both conditions $C \geqslant 0$, $C \neq 0$ are compatible; see, e.g., [20], [23]. The principal result of Part I will be an inequality corresponding to (1.2) but where $\|A\|$ is replaced by $(\frac{1}{2})$ meas $\operatorname{sp}(A)$, where $\operatorname{sp}(A)$ denotes the spectrum of A.

In Part II there will be considered a related problem concerning perturbations of a self-adjoint operator A. It will be supposed first (Theorem 2) that A and B are unitarily equivalent bounded self-adjoint operators whose difference D is semi-definite, so that

$$D = A - B \geqslant 0 \text{ (or } \leqslant 0) \text{ and } B = UAU^* \quad (U \text{ unitary}). \tag{1.3}$$

(1) This research was supported by the United States Air Force through the Air Force Office of Scientific Research of the Air Research and Development Command, under Contract No. AF 18 (603)–139. Reproduction in whole or in part is permitted for any purpose of the United States Government.