A FIXED POINT THEOREM IN SYMPLECTIC GEOMETRY

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There are a number of fixed point theorems peculiar to symplectic geometry. A particularly simple example is the theorem that any area-preserving mapping ψ of the two-dimensional sphere into itself possesses at least two distinct fixed points (see [6, 8]) although an arbitrary orientation-preserving mapping may have only one single fixed point. In higher dimensions such global theorems are not available, but it is known (see [11]) that any symplectic map ψ which is C^1-close to the identity map of a simply connected, compact symplectic manifold into itself has at least two fixed points. These fixed points are found as critical points of appropriate functions on the manifold. In this note we will derive a generalization of such a perturbation theorem which has various applications in mechanics.

To formulate our result we need some concepts of symplectic geometry: A smooth manifold Σ is called symplectic if there exists a non-degenerate closed 2-form ω on Σ; the symplectic manifold consists in fact of the pair (Σ, ω). If ω is even exact and given by $\omega = d\alpha$, α being a 1-form we call (Σ, α) an exact symplectic manifold. The most familiar example of an exact symplectic manifold is the cotangent bundle of any manifold with its natural 1-form.

A differentiable mapping ψ of Σ into itself is called symplectic if it preserves the two-form ω, i.e. if $\psi^*\omega = \omega$. Similarly, we call a mapping ψ exact symplectic if (Σ, α) is exact and $\psi^*\alpha - \alpha$ is exact, i.e. $= dF$ where F is a function of Σ. We apply the same terminology for mappings ψ of an open set $D_1 \subset \Sigma$ into another $D_2 \subset \Sigma$.

Of course, every exact symplectic mapping is also symplectic since $\psi^*\alpha = \alpha + dF$ implies

$$\psi^*\omega = d(\psi^*\alpha) = d\alpha = \omega.$$

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